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Decomposition of Balanced Matrices.
Part III: Parachutes

Michele Conforti[1]
Gérard Cornuéjols[2]
and
M.R. Rao[3]

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[1]Dipartimento di Matematica Pura ed Applicata, Università di Padova, Via Belzoni 7, 35131 Padova, Italy.

[2]Carnegie Mellon University, Schenley Park, Pittsburgh, PA 15213.

[3]New York University, 100 Trinity Place, New York, NY 10006.

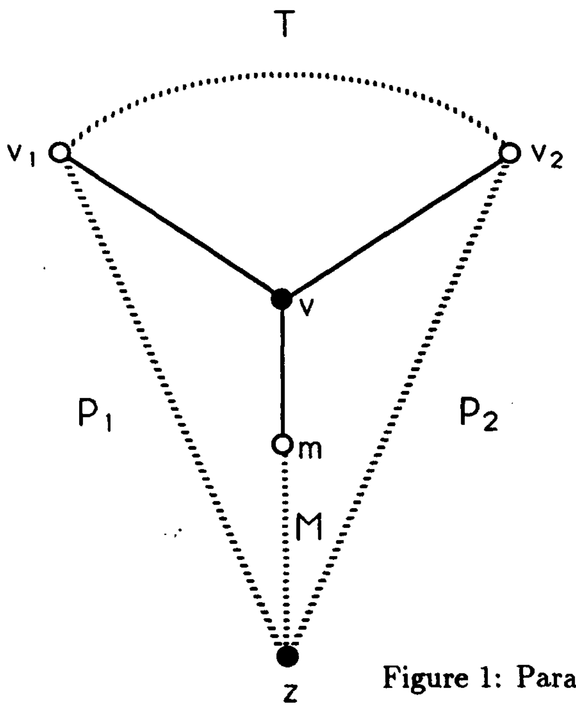


Figure 1: Parachute

1 Introduction

In this part, we consider wheel-free bipartite graphs G which are signable to be balanced and contain a parachute configuration. A parachute Π, denoted by $\Pi = Par(P_1, P_2, M, T)$, has *side paths* $P_1 = v_1, \ldots, z$ and $P_2 = v_2, \ldots, z$ where $|P_1|+|P_2| \geq 3$, *top path* $T = v_1, \ldots, v_2$ and *middle path* $M = v, m, \ldots, z$ where v is adjacent to nodes v_1 and v_2. See Figure 1.

The node z is called *bottom node*, v_1 and v_2 are called *side nodes* and v is called *center node*. We assume w.l.o.g. that $v \in V^c$. It follows that $v_1, v_2 \in V^r$ and $z \in V^c$. The nodes of $V(\Pi) \setminus \{v, v_1, v_2, m\}$ induce two connected components called the *top* of Π, induced by $V(T) \setminus \{v_1, v_2\}$, and the *bottom* of Π, induced by $V(P_1) \cup V(P_2) \cup V(M) \setminus \{v, v_1, v_2, m\}$.

Recall from Part I that, for a path $P = x_1, x_2, \ldots, x_{n-1}, x_n$, we denote by $\tilde{P}$ the subpath of P joining x_2 to x_{n-1}, i.e. $V(\tilde{P}) = V(P) \setminus \{x_1, x_n\}$. With this notation, the top of Π is $\tilde{T}$ and the bottom of Π is induced by $V(\tilde{P}_1) \cup V(\tilde{P}_2) \cup (V(M) \setminus \{v, m\})$.

When $|E(T)| = 2$, the parachute Π is said to have a *short top*; the top is *long* when $|E(T)| \geq 4$. Similarly, the parachute Π is said to have a *short middle* when $|E(M)| = 2$, and *long middle* otherwise. Finally, when $|E(P_1)| = 1$ or $|E(P_2)| = 1$, the parachute Π is said to have one *short side*; otherwise, we say that Π has *long sides.*

This part is organized as follows. In Section 2, we list all possible strongly adjacent nodes to a parachute Π. In Sections 3 and 4, we list all possible direct connections from the top of Π to the bottom, avoiding $N(v) \cup (N(v_1) \cap N(v_2))$. When no such path exists, the graph G has an extended star cutset disconnecting the top of Π from the bottom. When there is such a path, at least one of the following possibilities arises.

- The graph G contains no parachute with long sides. This case is treated in Section 5 where we prove the existence of an extended star cutset disconnecting G.
- The graph G contains a stabilized parachute. This concept is defined in Section 6 where an extended star cutset is shown to disconnect G.
- The graph G contains a parachute with short middle path and long sides, but G contains no stabilized parachute and no connected squares. This case is treated in Section 7 where we prove the existence of an extended star cutset.
- The graph G contains connected squares. This case is treated in Part IV.
- The graph G contains goggles. This case is treated in Part V.

2 Strongly Adjacent Nodes

Theorem 2.1 *Let $\Pi = Par(P_1, P_2, M, T)$ be a parachute in a wheel-free bipartite graph G that is signable to be balanced. Let $w \in V(G) \setminus V(\Pi)$ be a strongly adjacent node to Π. Then w satisfies one of the following properties.*
(i) w has exactly two neighbors in Π and both are in $V(P_1)$ or in $V(P_2)$ or in $V(M)$ or in $V(T)$,
(ii) w is of one of the following types.

- **Type a** *Node $w \in V^c$ is adjacent to the neighbors of z in P_1 and P_2 respectively and to no other node of Π.*
- **Type b** *Node $w \in V^r$ is adjacent to one node in $V(\tilde{P}_1)$, to one node in $V(\tilde{P}_2)$ and to no other node of Π.*

- **Type c** *Node* $w \in V^c$ *is adjacent to exactly two nodes of* Π, *one of which is the neighbor of* z *in* $\tilde{M}$ *and the other is the neighbor of* z *in* $\tilde{P}_1$ *or in* $\tilde{P}_2$.

- **Type d** *Node* $w \in V^r$ *is adjacent to one node in* $V(M) \setminus \{z\}$, *to one node in either* $V(\tilde{P}_1)$ *or* $V(\tilde{P}_2)$ *(but not both) and to no other node of* Π.

- **Type e** *Node* $w \in V^r$ *is adjacent to* v, *to one node in* $V(\tilde{T})$ *and to no other node of* Π.

- **Type f** *Node* $w \in V^c$ *is adjacent to one node in* $V(P_1)$, *to one node in* $V(P_2)$, *to one node of* $V(\tilde{M})$ *and to no other node of* Π.

- **Type g** *Node* $w \in V^c$ *is adjacent to* m, *to two nodes in* $V(T)$ *and to no other node of* Π.

- **Type h** *Node* $w \in V^r$ *is adjacent to* v, *to one node in* $V(\tilde{T})$, *one node in* $V(M) \setminus \{v\}$ *and to no other node of* Π.

- **Type i** *Node* $w \in V^r$ *is adjacent to* v, *to one node in* $V(\tilde{T})$, *to one node in either* $V(\tilde{P}_1)$ *or* $V(\tilde{P}_2)$ *(but not both) and to no other node of* Π.

When Π *has a short side, say* P_2, *the following additional types of strongly adjacent nodes can occur.*

- **Type j** *Node* $w \in V^c$ *is adjacent to* v_2, *to one node in* $V(\tilde{M})$ *and to no other node of* Π.

- **Type k** *Node* $w \in V^c$ *is adjacent to one node in* $v(T) \setminus \{v_1\}$, *to one node in* $V(\tilde{P}_1)$ *and to no other node of* Π.

- **Type l** *Node* $w \in V^r$ *is adjacent to the neighbors of* v_1 *in* $v(T)$ *and* $V(P_1)$ *respectively and to no other node of* Π.

- **Type m** *Node* $w \in V^r$ *is adjacent to two nodes of* $V(M) \setminus \{v\}$, *to the neighbor of* v_2 *on* $V(T)$ *and to no other node of* Π.

- **Type n** *Node* $w \in V^c$ *is adjacent to* v_2, *to one node in* $V(\tilde{T})$, *to one node in* $V(\tilde{M}) \setminus \{m\}$ *and to no other node of* Π.

When Π *has a short top and long sides, one additional type of strongly adjacent node can occur.*

- **Type o** *Node* $w \in V^r$ *is adjacent to two nodes of* $V(M) \setminus \{v\}$*, to the unique node of* $V(\tilde{T})$ *and to no other node of* Π.

Proof: In this proof, we assume that the short side, if any, is P_2. We first show that w cannot have more that three neighbors in Π. Assume the contrary. Then w has at least two neighbors in $V(M)\setminus\{z\}$, else w has three or more neighbors in $V(P_1) \cup V(P_2) \cup V(T)$, contradicting the assumption that G is wheel-free. Since w cannot have three or more neighbors in $V(P_1) \cup V(M)$ and $V(P_2) \cup V(M)$, it follows that w has no neighbor in $V(P_1) \cup V(P_2)$ and exactly two neighbors in $V(M)$. Since w cannot have three or more neighbors in $V(T) \cup \{v\}$, this implies that w has two neighbors in $V(\tilde{T})$ and is not adjacent to v. The nodes of $V(\Pi) \setminus V(\tilde{P}_1)$ induce a cycle with unique chord vv_2 and w is strongly adjacent to it. Since w is not of Type 1, 2 or 3 of Theorem I.3.3, we conclude that w has at most three neighbors in Π.

Now, we divide the proof into the cases where w has two or three neighbors in Π.

Case 1 Node w has two neighbors w_1, w_2 in Π.

If both nodes belong to $V(P_1)$ or to $V(P_2)$ or to $V(T)$ or to $V(M)$, then we are in Case (i) of the theorem. Now, we enumerate the other possibilities.

Case 1.1 Node $w_1 \in V(\tilde{P}_1)$ and $w_2 \in V(\tilde{P}_2)$.

If $w \in V^c$, then both w_1 and w_2 must be adjacent to z otherwise, if say $w_1 \in V^r$ is not adjacent to z, there exists a $3PC(z, w_1)$. Hence, w is of Type a. If $w \in V^r$, then w is of Type b.

Case 1.2 Node $w_1 \in V(\tilde{P}_1)$ and $w_2 \in V(M) \setminus \{z\}$.

If $w \in V^c$, then both w_1 and w_2 must be adjacent to z, otherwise there is a $3PC(z, w_1)$ or a $3PC(z, w_2)$. Hence w is of Type c. If $w \in V^r$, then w is of Type d.

Case 1.3 Node $w_1 \in V(\tilde{P}_1)$ and $w_2 \in V(T) \setminus \{v_1\}$. The side path P_2 is short since otherwise, there is a $3PC(z, v_2)$. If $w \in V^c$, then w is of Type k. If $w \in V^r$, then both w_1 and w_2 are adjacent to v_1, else there exists a $3PC(v_1, w_1)$ or a $3PC(v_1, w_2)$. Hence w is of Type l.

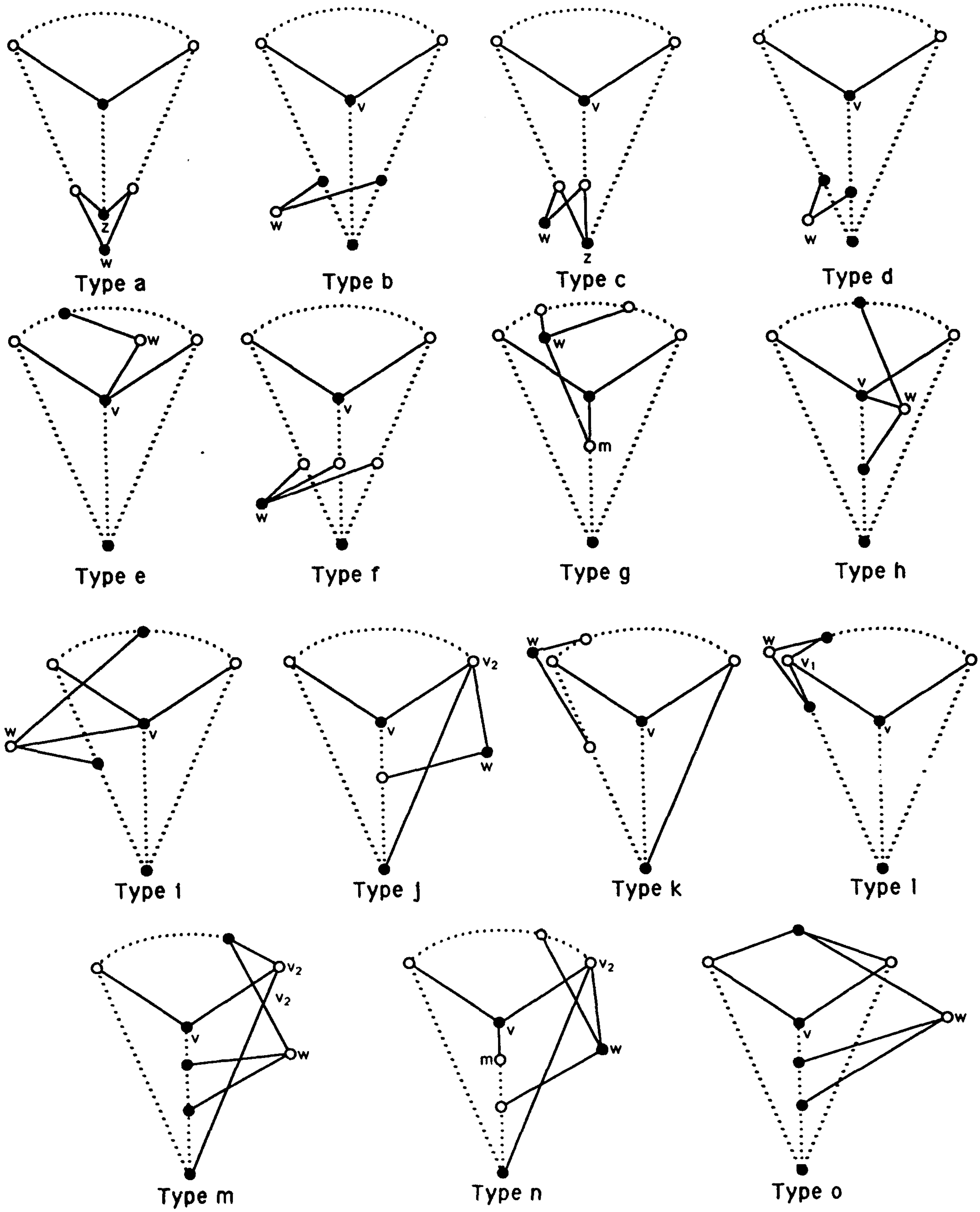


Figure 2: Strongly Adjacent Nodes

Case 1.4 Node $w_1 \in V(T)$ and $w_2 \in V(M)$.

Assume $w \in V^r$. Then $w_2 = v$, else there exists a $3PC(v_1, w_2)$ since we have assumed that the short side, if any, is P_2. Hence, w is of Type e.

Assume $w \in V^c$. Then $w_1 = v_1$ or v_2, else there exists a $3PC(v, w_1)$. Assume w.l.o.g. that $w_1 = v_2$. Then P_2 is short, else there exists a $3PC(z, v_2)$. Hence w is of Type j.

By symmetry, the four above subcases exhaust all the possibilities for Case 1.

Case 2 Node w has three neighbors w_1, w_2, w_3 in Π.

The nodes w_1, w_2, w_3 cannot all belong to any of the sets $V(P_1) \cup V(\tilde{M})$, $V(P_2) \cup V(M)$, $V(T) \cup \{v\}$, $V(P_1) \cup V(P_2) \cup V(T)$, otherwise there exists a wheel. This leaves the following possibilities.

Case 2.1 Node $w_1 \in V(P_1) \setminus \{z\}$, $w_2 \in V(M) \setminus \{z\}$ and $w_3 \in V(P_2) \setminus \{z\}$.

Assume $w \in V^r$. If $w_2 \neq v$, there exists a $3PC(w, v)$ and if $w_2 = v$, there exists an odd wheel with center v. Assume $w \in V^c$. Then w is of Type f.

Case 2.2 Node $w_1 \in V(\tilde{P}_1)$, $w_2 \in V(M) \setminus \{z\}$ and $w_3 \in V(T) \setminus \{v_1\}$.

If $w \in V^c$, there exists a $3PC(w, v_1)$. If $w \in V^c$ and $w_2 \neq v$, there exists a $3PC(v_2, w_2)$. So w is of Type i.

Case 2.3 Nodes $w_1, w_2 \in V(M)$ and $w_3 \in V(\tilde{T})$. If $w \in V^c$, there is a $3PC(w_3, v)$. So $w \in V^r$. Let w_1 be the neighbor of w which is closest to v in M.

If $w_1 = v$, then w is of Type h.

If $w_1 \neq v$ and w_3 is not adjacent to v_1, then P_2 is short, otherwise there exists a $3PC(v_1, w_3)$. Furthermore, w_3 is adjacent to v_2, else there exists a $3PC(v_2, w_3)$. Hence w is of Type m.

If w_3 is adjacent to both v_1 and v_2, then the top path is short and node w is of Type m or o depending on whether Π has a short side or not.

Case 2.4 Nodes $w_1, w_2 \in V(T)$ and $w_3 \in V(\tilde{M})$.

If $w \in V^r$, there exists a $3PC(v,w)$. So $w \in V^c$. Let w_1 be the neighbor of w which is closest to v_1 in T. If $w_3 = m$, node w is of Type g. Now we assume $w_3 \neq m$. If $w_1 = v_1$ and $w_2 = v_2$, then w is of Type f. If $w_1 \neq v_1$ then P_2 is short, else there exists a $3PC(z, v_2)$. Furthermore $w_2 = v_2$, else there exists a $3PC(z, v_1)$. Then w is of Type n. $\square$

Corollary 2.2 *In a wheel-free balanced bipartite graph, all strongly adjacent nodes described in Theorem 2.1 can exist, except for a Type b[2.1] node w having neighbors $b_1 \in V(\tilde{P}_1)$ and $b_2 \in V(\tilde{P}_2)$ adjacent to v_1 and v_2 respectively.*

3 Parachute Modifications

Let $\Pi = Par(P_1, P_2, M, T)$ be a parachute with center node $v \in V^c$ and side nodes v_1, v_2. If Π has long top, let $S(\Pi) = N(v) \cup (N(v_1) \cap N(v_2))$. If Π has short top, let t be the unique node of $V(\tilde{T})$ and let $S(\Pi) = N(v) \cup (N(v_1) \cap N(v_2)) \setminus \{t\}$. In this section and in the next one, we enumerate all possible direct connections from the top of Π to the bottom, avoiding $S(\Pi)$ (the definition of a direct connection can be found in Part I).

Let $Q = x_1, \ldots, x_n$ denote a direct connection avoiding $S(\Pi)$, where x_1 is adjacent to $V(\tilde{P}_1) \cup V(\tilde{P}_2) \cup (V(M) \setminus \{v, m\})$ and x_n is adjacent to $V(\tilde{T})$. It follows from the definition of a direct connection that, for $2 \leq j \leq n-1$, the node x_j is not adjacent to $V(\Pi) \setminus \{v_1, v_2, m\}$. Furthermore, since Q avoids $S(\Pi)$, node x_j is adjacent to at most one of the two nodes v_1, v_2. To reduce the number of possible path types that need to be enumerated in the main theorem of this section (Theorem 3.4), we introduce the concept of *parachute modification.*

Definition 3.1 *Let $\Pi = Par(P_1, P_2, M, T)$ be a parachute with center node $v \in V^c$, bottom node z and side nodes v_1, v_2.*

Parachute modifications at the top *are defined as follows.*

Type 1 *Assume $y \in V(G) \setminus V(\Pi)$ has exactly two neighbors in Π, both are in $V(T)$ and at least one is in $\tilde{T}$. A* parachute modification of Type 1 at the top *consists of replacing Π by the unique parachute Π' which is induced by a subset of $V(\Pi) \cup \{y\}$ and is distinct from Π.*

Type 2 *For $k \geq 2$ and $y_j \in V(G) \setminus V(\Pi)$, $1 \leq j \leq k$, assume $y_1, \dots, y_k$ is a chordless path such that*

(i) Node y_1 is adjacent to either v_1 or v_2, say v_1, and to no other node of Π.

(ii) $y_k \in V^c$ is adjacent to one node of $V(T) \setminus \{v_1\}$ and to no other node of Π.

(iii) For $2 \leq j \leq k-1$, node y_j has no neighbor in Π.

A parachute modification of Type 2 at the top *consists of replacing Π by the unique parachute Π' which is induced by a subset of $V(\Pi) \cup \{y_1, \dots, y_k\}$ and is distinct from Π.*

Parachute modifications at the bottom *are defined as follows.*

Type 1 *Assume $y \in V(G) \setminus V(\Pi)$ has exactly two neighbors in Π and both are in $V(P_1)$ or $V(P_2)$ or $V(M) \setminus \{v\}$. A* parachute modification of Type 1 at the bottom *consists of replacing Π by the unique parachute Π' which is induced by a subset of $V(\Pi) \cup \{y\}$ and is distinct from Π.*

Type 2 *Assume $y \in V^c$ is a strongly adjacent node of Type f[2.1] with neighbors $n_1 \in V(P_1)$, $n_2 \in V(P_2)$ and $n_3 \in V(\tilde{M})$. Furthermore assume that, if $n_1 \equiv v_1$ then P_1 is short and if $n_2 \equiv v_2$ then P_2 is short. A* parachute modification of Type 2 at the bottom *consists of replacing Π by the unique parachute Π' which is induced by a subset of $V(\Pi) \cup \{y\}$ and has bottom node y.*

Type 3 *Assume w.l.o.g that the side path P_1 of Π is long. For $k \geq 2$ and $y_j \in V(G) \setminus V(\Pi)$, $1 \leq j \leq k$, assume $y_1, \dots, y_k$ is a chordless path such that*

(i) Node y_1 is adjacent to v_1 and to no other node of Π.

(ii) Node $y_k \in V^c$ is adjacent to one node of $V(\tilde{P}_1)$ and to no other node of Π, or y_k is a strongly adjacent node of Type c or j[2.1], adjacent to nodes in M and P_2.

(iii) For $2 \leq j \leq k-1$, node y_j has no neighbor in Π.

A parachute modification of Type 3 at the bottom *consists of replacing Π by the unique parachute Π' which is induced by a subset of $V(\Pi) \cup \{y_1, \dots, y_k\}$, has top path T and is distinct from Π.*

Type 4 *For $k \geq 2$ and $y_j \in V(G) \setminus V(\Pi)$, $1 \leq j \leq k$, assume $y_1, \ldots, y_k$ is a chordless path such that*

(i) y_1 is adjacent to m and to no other node of Π.

(ii) $y_k \in V^c$ is adjacent to one node of $V(\tilde{M}) \setminus \{m\}$ and to no other node of Π, or y_k is a strongly adjacent node of Type a[2.1].

(iii) For $2 \leq j \leq k-1$, node y_j has no neighbor in Π.

A parachute modification of Type 4 at the bottom *consists of replacing Π by the unique parachute Π' which is induced by a subset of $V(\Pi) \cup \{y_1, \ldots, y_k\}$, has top path T and is distinct from Π.*

Remark 3.2 *(i) If Π is a parachute with long top, then Π' obtained from Π by parachute modification has also long top.*

(ii) If Π is a parachute with long sides, then Π' obtained from Π by parachute modification has also long sides.

Let Π be a parachute and $Q = x_1, \ldots, x_n$ a direct connection from bottom to top avoiding $S(\Pi)$. Assume $n \geq 2$. A *parachute modification relative to $V(Q)$* is a parachute modification of Π which only involves the nodes of $V(\Pi) \cup V(Q)$.

Theorem 3.3 *Let G be a wheel-free bipartite graph that is signable to be balanced. Let $\Pi = Par(P_1, P_2, M, T)$ be a parachute and $Q = x_1, \ldots, x_n$ be a direct connection from bottom to top avoiding $S(\Pi)$ such that no parachute modification exists relative to $V(Q)$.*

(i) If Π has long top and long sides, then $n \geq 2$ and Q is of one of the following types.

- **Type a** *Node x_1 is a strongly adjacent node to Π, adjacent to v_1, m and some node $b \in V(\tilde{P}_2)$. Node x_n is not strongly adjacent to Π and its unique neighbor $t \in V(\tilde{T})$ is adjacent to v_1. Exactly one of the nodes x_j, for $2 \leq j \leq n-1$, is adjacent to m and none is adjacent to v_1, v_2.*

- **Type b** *Node x_n is adjacent to v_2, m and some node $t \in V(\tilde{T})$. Node x_1 is not strongly adjacent to Π and its unique neighbor $b \in V(\tilde{P}_2)$ is adjacent to v_2. For $2 \leq j \leq n-1$, x_j has no neighbor in Π. Furthermore, Π has a short middle path.*

- **Type c** *Node* x_1 *is adjacent to* v_2, m *and some node* $b \in V(\tilde{P}_1)$. *Node* x_n *is adjacent to* v_2, m *and some node* $t \in V(\tilde{T})$. *For* $2 \leq j \leq n-1$, x_j *has no neighbor in* Π.

- **Connected 6-hole** *Node* x_1 *is a Type b[2.1] node having neighbors* $b_1 \in V(\tilde{P}_1)$ *and* $b_2 \in V(\tilde{P}_2)$ *adjacent to* v_1 *and* v_2 *respectively. Node* x_n *is not strongly adjacent and its neighbor* $t \in \tilde{T}$ *belongs to* V^r. *For* $2 \leq j \leq n-1$, x_j *has no neighbor in* Π.

(ii) If Π *has short top and long sides, then either* $n = 1$ *and the only node of* Q *is of Type o[2.1], or* $n \geq 2$ *and* Q *is of Type a (this theorem) or is of Types d, e or f described below.*

- **Type d** *Node* $x_1 \in V^c$ *is not strongly adjacent to* Π *and its unique neighbor belongs to* $V(\tilde{M}) \setminus \{m\}$. *Node* x_n *is not strongly adjacent to* Π. *For* $2 \leq j \leq n-1$, x_j *has no neighbor in* Π.

- **Type e** *Node* $x_1 \in V^r$ *is not strongly adjacent to* Π *and its unique neighbor, say* b, *belongs to* $V(\tilde{M}) \setminus \{m\}$. *Node* x_n *is not strongly adjacent to* Π. *Node* m *is adjacent to* b *and to exactly one of the nodes* x_j, *for* $2 \leq j \leq n-1$. *Nodes* v_1, v_2 *are not adjacent to* $V(Q)$.

- **Type f** *Node* x_1 *is a strongly adjacent node of Type a[2.1]. Node* x_n *is not strongly adjacent to* Π. *For* $2 \leq j \leq n-1$, x_j *has no neighbor in* Π.

(iii) If Π *has a short side, say* P_2, *then either* $n = 1$ *and the only node of* Q *is of Type k, l, m or n[2.1], or* $n \geq 2$ *and* Q *is of Type b (this theorem) or is of Types g, h, i, j, k, l, m, n, o, p or q described below.*

- **Type g** *Nodes* $x_1, x_n \in V^r$ *are not strongly adjacent to* Π *and their respective neighbors* $b \in V(\tilde{P}_1)$ *and* $t \in V(\tilde{T})$ *are adjacent to* v_1. *For* $2 \leq j \leq n-1$, x_j *has no neighbor in* Π.

- **Type h** *Nodes* $x_1, x_n \in V^c$ *are not strongly adjacent to* Π *and their neighbors belong to* $V(\tilde{P}_1)$ *and* $V(\tilde{T})$ *respectively. For* $2 \leq j \leq n-1$, x_j *has no neighbor in* Π.

- **Type i** *Nodes* $x_1 \in V^r$ *and* $x_n \in V^c$ *are not strongly adjacent to* Π, *the neighbor of* x_1 *belongs to* $V(\tilde{M}) \setminus \{m\}$ *and the neighbor of* x_n *in* $V(\tilde{T})$ *is adjacent to* v_2. *For* $2 \leq j \leq n-1$, x_j *has no neighbor in* Π.

- **Type j** *Node* x_1 *is a strongly adjacent node of Type j[2.1] and* $x_n \in V^r$ *is not strongly adjacent to* Π. *For* $2 \leq j \leq n-1$, x_j *has no neighbor in* Π.

- **Type k** *Node* x_n *is not strongly adjacent to* Π *and the neighbor of* x_n *in* $V(\hat{T})$ *is adjacent to* v_2. *Node* $x_1 \in V^r$ *is not strongly adjacent to* Π *and its unique neighbor, say b, belongs to* $V(\tilde{M}) \setminus \{m\}$. *Node m is adjacent to b and to exactly one of the nodes* x_j, *for* $2 \leq j \leq r-1$. *Nodes* v_1, v_2 *are not adjacent to* $V(Q)$.

- **Type l** *Node* x_n *is a strongly adjacent node of Type g[2.1]. Node* $x_1 \in V^r$ *is not strongly adjacent to* Π *and its unique neighbor, say b, belongs to* $V(\tilde{M})$ *and is adjacent to m. For* $2 \leq j \leq n-1$, x_j *has no neighbor in* Π.

- **Type m** *Node* x_n *is a strongly adjacent node of Type g[2.1]. Node* $x_1 \in V^c$ *is not strongly adjacent to* Π *and its unique neighbor, say b, belongs to* $V(\tilde{P}_1)$ *and is adjacent to z. For* $2 \leq i < j \leq n-1$, x_i *is adjacent to m,* x_j *is adjacent to* v_1 *and no other adjacencies exist between* $V(Q)$ *and* $V(\Pi)$. *Furthermore* Π *has short middle path.*

- **Type n** *Node* x_n *is a strongly adjacent node of Type g[2.1]. Node* $x_1 \in V^c$ *is not strongly adjacent to* Π *and its unique neighbor, say b, belongs to* $V(\tilde{P}_1)$. *For* $2 \leq j \leq i \leq n-1$, x_i *is adjacent to m,* x_j *is adjacent to* v_1 *and no other adjacencies exist between* $V(Q)$ *and* $V(\Pi)$.

- **Type o** *Nodes* $x_1, x_n \in V^c$ *are not strongly adjacent to* Π *and their neighbors belong to* $V(\tilde{P}_1)$ *and* $V(\hat{T})$ *respectively. Node* v_2 *is adjacent to two nodes of* $V(Q)$, *say* x_i *and* x_k. *Node m is adjacent to node* x_i *and to another node of* $V(Q)$, *say* x_l, *where* $k \leq l < i$. *Node* v_1 *is not adjacent to* $V(Q)$.

- **Type p** *Nodes* $x_1 \in V^r$ *and* $x_n \in V^c$ *are not strongly adjacent to* Π. *The unique neighbor of* x_1 *in* Π *belongs to* $V(\tilde{M})$ *and is adjacent to m. One of the nodes* x_j, *for* $2 \leq j \leq n-1$, *is adjacent to m and to* v_2. *The other nodes of* $V(Q)$ *are not adjacent to* Π.

- **Type q** *Nodes* $x_1 \in V^c$ *and* $x_n \in V^r$ *are not strongly adjacent to* Π. *The unique neighbor of* x_1 *in* Π *belongs to* $V(\tilde{P}_1)$ *and the unique* *neighbor of x_n is adjacent to v_2. One of the nodes x_j, for $2 \leq j \leq n-1$, is adjacent to v_2. Two nodes x_i, x_k, for $j \leq k < i \leq n-1$ are adjacent to m. The other nodes of $V(\tilde{Q})$ are not adjacent to* Π.

Proof: First, we consider the case $n = 1$, i.e. Q consists of a single node which is strongly adjacent to Π. Then it follows from Theorem 2.1 that this node is of Type k, l, m, n or o[2.1].

Now consider the case $n \geq 2$. By Theorem 2.1, either x_n is not strongly adjacent to Π or it is a strongly adjacent node of Type g[2.1]. Similarly, either x_1 is not strongly adjacent to Π or it is a strongly adjacent node of Type a, b, c, d, f or j[2.1]. We will divide the proof into two parts, depending on whether x_n is of Type g[2.1] or is not strongly adjacent to Π. Then, in each of the two parts, the proof will be broken down further based on the adjacencies between $\{x_2, \ldots, x_{n-1}\}$ and $\{v_1, v_2, m\}$. Finally, subcases will occur depending on the type of node x_1. The two following claims reduce the number of cases that have to be considered.

We say that node $x_i \in V(Q)$ adjacent to m but not v_1 or v_2 and node $x_j \in V(Q)$ adjacent to v_1 or v_2 but not m are *consecutive* in Q if no intermediate node of the $x_i x_j$-subpath of Q is adjacent to at least one node in $\{m, v_1, v_2\}$. When $x_i = x_j$, we also say x_i and x_j are consecutive in Q.

Claim 1 If $x_i, x_j \in V(\tilde{Q})$ are consecutive in Q, where x_i is adjacent to m and x_j is adjacent to v_2, then P_2 is short.

Proof of Claim 1: Let Q_{ij} be the $x_i x_j$-subpath of Q. Now $V(Q_{ij}) \cup V(T) \cup V(P_1) \cup V(M)$ induces an odd wheel with center v unless P_2 is short. This proves Claim 1.

Claim 2 There does not exist nodes $x_i, x_j \in V(\tilde{Q})$ such that x_i is adjacent to v_1 and x_j is adjacent to v_2.

Proof of Claim 2: Choose nodes $x_i, x_j \in V(\tilde{Q})$ such that x_i is adjacent to v_1, x_j is adjacent to v_2 and the subpath Q_{ij} of Q connecting them is shortest. The length of the path Q_{ij} is at least 2, since $x_i = x_j$ would imply that $x_i \in S(\Pi)$, a contradiction to the definition of Q. First assume that neither x_i nor x_j is adjacent to m. If no intermediate node of Q_{ij} is adjacent to m, there exists a parachute modification of Type 2 at the top. Otherwise there exist (possibly coincident) nodes x_k, x_l adjacent to m such that x_i and x_k are consecutive in Q and x_j and x_l are consecutive in Q. By Claim 1, this implies that both P_1 and P_2 are short, a contradiction. Now assume that x_j

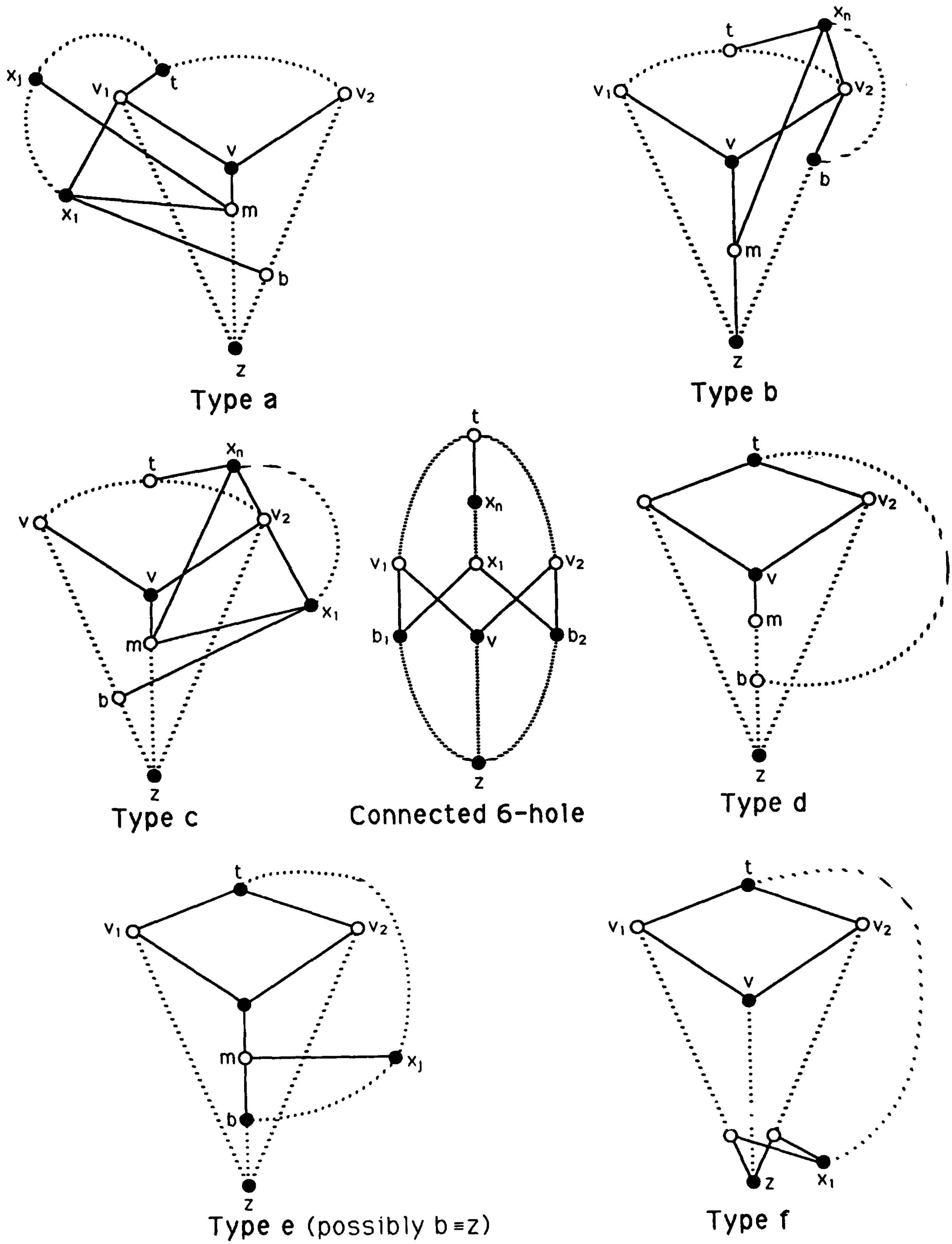

x_j
t
v_1
v_2
v
m
x_1
b
z
Type a
t
x_n
v_1
v_2
v
b
m
z
Type b
t
x_n
v
v_2
v
x_1
m
b
z
Type c
t
x_n
v_1
x_1
v_2
b_1
v
b_2
z
Connected 6-hole
t
v_2
v
m
b
z
Type d
t
v_1
v_2
m
x_j
b
z
Type e (possibly b≡z)
t
v_1
v_2
v
z
x_1
Type f

Figure 3

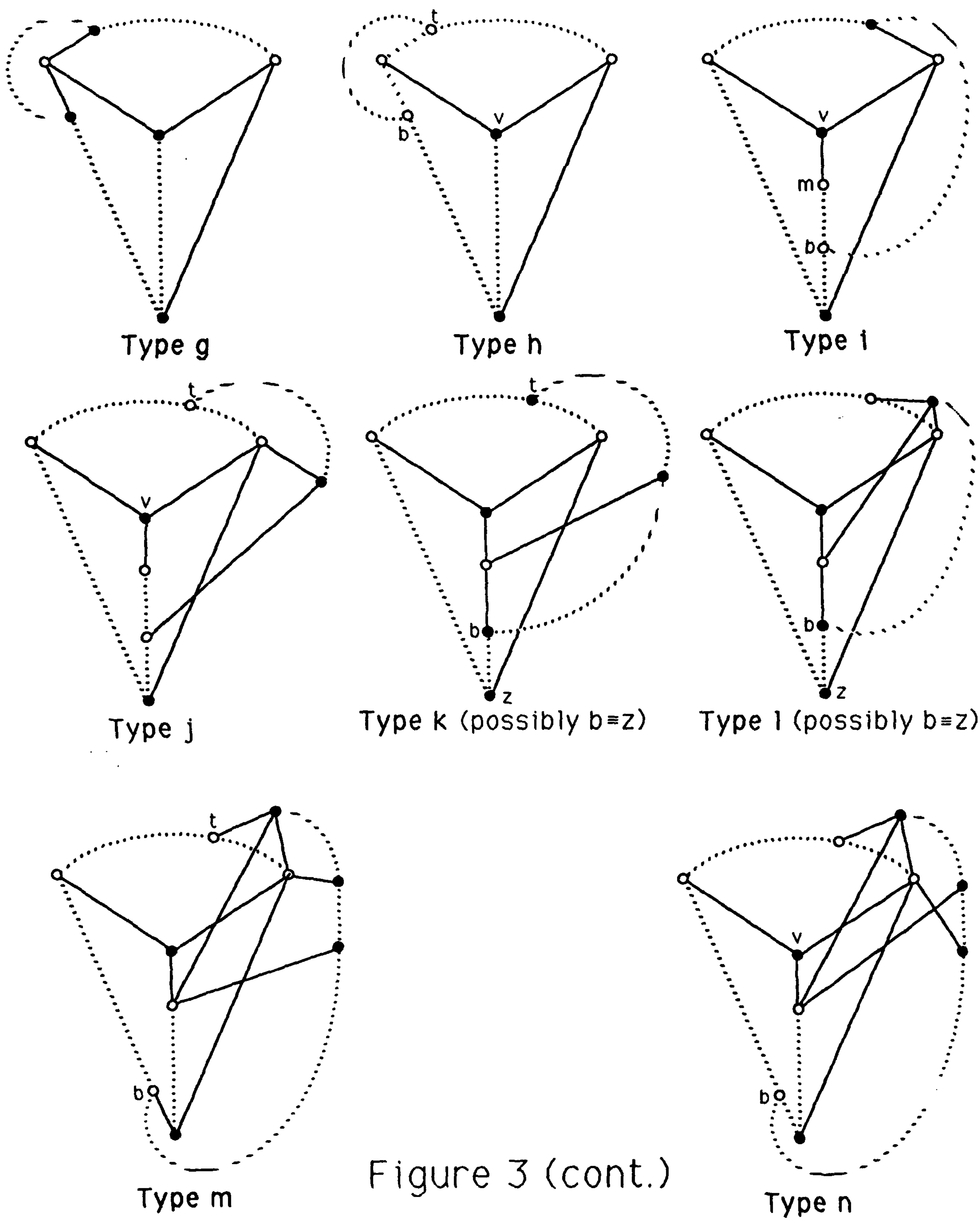
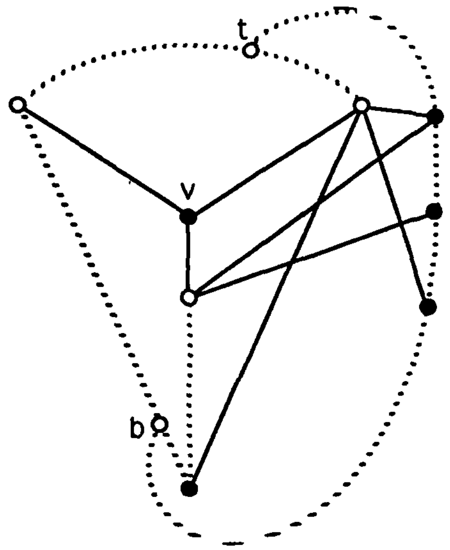
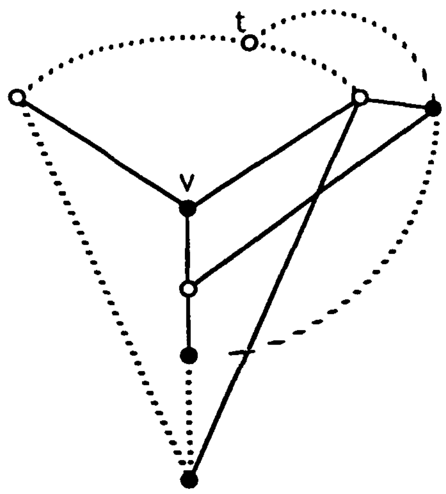


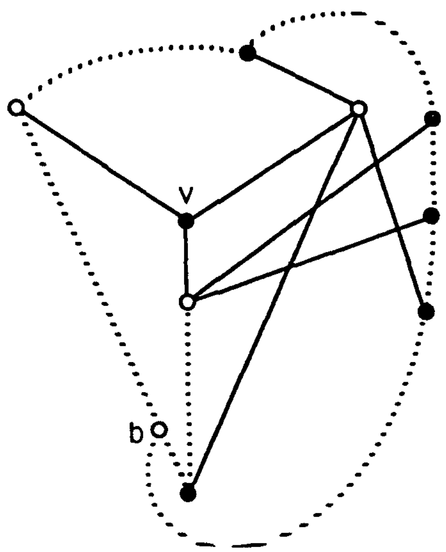
Figure 3 (cont.)



Type o



Type p



Type q

Figure 3 (cont.)

is adjacent to m. Since x_j is also adjacent to v_2, it follows that P_2 is short. Therefore P_1 is long and x_i is not adjacent to m. Furthermore, by Claim 1, no intermediate node of Q_{ij} is adjacent to m. This implies the existence of a parachute modification of Type 3 at the bottom. This proves Claim 2.

Part 1 Node x_n is of Type g[2.1]

It follows from the definition of Q that x_n is not adjacent to both v_1 and v_2. Assume w.l.o.g. that x_n is not adjacent to v_1. Let T_1 and T_2 be the chordless paths from x_n to v_1 and from x_n to v_2 which only use nodes of $V(T) \cup \{x_n\}$. Let $P_1 \equiv T_1$, $P_2 = T_2, v, v_1$ and P_3 be any chordless path from x_n to v_1 with nodes in $V(Q) \cup V(P_1) \cup V(\tilde{P}_2) \cup V(\tilde{M}) \setminus \{m\}$. Since G is signable to be balanced, the paths P_1, P_2, P_3 do not form a $3PC(x_n, v_1)$. This implies that v_2 is adjacent to at least one node of $V(\tilde{P}_3)$. Now let $P_2' = x_n, m, v, v_1$. Since the paths P_1, P_2', P_3 do not form a $3PC(x_n, v_1)$, node m is adjacent to at least one node of $V(\tilde{P}_3)$. It follows from Claims 1 and 2 that $\tilde{Q}$ contains no node adjacent to v_1.

Case 1 $N(V(\tilde{Q})) \cap \{v_1, v_2, m\} = \emptyset$.

Case 1.1 Node x_1 is not strongly adjacent to Π.

Let b be the node of Π adjacent to x_1. Since both v_2 and m are adjacent to $V(\tilde{P}_3)$, it follows that either b is adjacent to v_2 and m is adjacent to z, or b is adjacent to m and v_2 is adjacent to z. Furthermore, in both cases, node x_n is adjacent to v_2, else there is a $3PC(x_n, v_2)$. This yields paths Q of Types b and l respectively.

Case 1.2 Node x_1 is strongly adjacent to Π.

Since v_2 and m must be adjacent to $V(\tilde{P}_3)$, it follows that x_1 is not of Type a, b, c or d[2.1]. If x_1 were of Type j[2.1], then it would have to be adjacent to both v_2 and m, contradicting the fact that x_1 is adjacent to a node of $V(\Pi) \setminus N(v)$. The last case to consider is when x_1 is of Type f[2.1] and is adjacent to v_2, m and a node $b \in V(\tilde{P}_1)$. Then we must have x_n is adjacent to v_2, else there is a $3PC(x_n, v_2)$. This yields a path Q of Type c.

Case 2 $N(V(\tilde{Q})) \cap \{v_1, v_2, m\} = \{v_2\}$.

Node x_n is adjacent to v_2, else there is a $3PC(x_n, v_2)$. Since there is no wheel, $\tilde{Q}$ contains exactly one neighbor of v_2.

Case 2.1 Node x_1 is not strongly adjacent to Π.

Let b be the node of Π adjacent to x_1. Since m must be adjacent to $\tilde{P}_3$, it follows that either $b \in V(\tilde{P}_2)$ and m is adjacent to z, or $b \in V(\tilde{M})$ is adjacent to m. If $b \in V(\tilde{P}_2)$ is adjacent to v_2, then there is a wheel with center v_2. If $b \in V(\tilde{P}_2)$ is not adjacent to v_2, then there is a parachute modification of Type 3 at the bottom relative to $V(Q)$, a contradiction. If b is adjacent to m, then either there is a $3PC(z, v_2)$ when P_2 is long, or there is a wheel with center v_2 when P_2 is short.

Case 2.2 Node x_1 is strongly adjacent to Π.

Since m must be adjacent to $\tilde{P}_3$, it follows that x_1 is not of Type a, b or d[2.1]. If x_1 is of Type c[2.1], the middle path M must be short, i.e. x_1 is adjacent to m. If x_1 is adjacent to m and to $b \in V(\tilde{P}_2)$, there is a $3PC(x_1, v_2)$. If x_1 is adjacent to m and to $b \in V(\tilde{P}_1)$, there is a parachute modification of Type 3 at the bottom relative to $V(Q)$, a contradiction to our choice of Π and Q. If x_1 is of Type j[2.1], then it cannot be adjacent to m by definition of Q, a contradiction. If x_1 is of Type f[2.1] and is adjacent to v_2, there is a wheel with center v_2. If x_1 is of Type f[2.1] and is not adjacent to v_2, there is a $3PC(x_1, v_2)$.

Case 3 $N(V(\tilde{Q})) \cap \{v_1, v_2, m\} = \{m\}$.

Since there is no wheel, $\tilde{Q}$ contains exactly one neighbor of m.

Case 3.1 Node x_1 is not strongly adjacent to Π.

Let b be the neighbor of x_1 in Π. If $b \in V(M) \setminus \{v, m\}$ is adjacent to m, there is a wheel with center m. If $b \in V(M) \setminus \{v, m\}$ is in V^c but is not adjacent to m, there is a $3PC(m, b)$. If $b \in V(M) \setminus \{v, m\}$ is in V^r, there is a parachute modification of Type 3 at the bottom. It follows that $b \in V(\tilde{P}_1) \cup V(\tilde{P}_2)$. Since v_2 must be adjacent to $\tilde{P}_3$, b is adjacent to v_2 and P_2 is long. Now, there is a $3PC(m, b)$.

Case 3.2 Node x_1 is strongly adjacent to Π.

If x_1 is of Type a[2.1], then there is a parachute modification of Type 4 at the bottom, relative to $V(Q)$, a contradiction. Since v_2 must be adjacent to $\tilde{P}_3$, it follows that x_1 is not of Type b, c or d (Theorem 2.1). If x_1 is of Type f or j[2.1], then either it is adjacent to m and there is an odd wheel with center m, or it is not and there is a $3PC(m, x_1)$.

Case 4 $N(V(\tilde{Q})) \cap \{v_1, v_2, m\} = \{v_2, m\}$.

Node x_n is adjacent to v_2, else there is a $3PC(x_n, v_2)$. Furthermore P_2 is short, by Claim 1. Let $x_j \in V(\tilde{Q})$ be the neighbor of v_2 closest to x_n in Q and $x_i \in V(\tilde{Q})$ the neighbor of m closest to x_n. Note that $i = j$ is possible.

Case 4.1 Node x_1 is not strongly adjacent to Π.

Let b be the neighbor of x_1 in Π. If $b \in V(M) \setminus \{v, m\}$, there is an odd wheel with center v_2. So $b \in V(\tilde{P}_1)$. This implies that x_j is the only neighbor of v_2 in $\tilde{Q}$ and x_i is the only neighbor of m in $\tilde{Q}$, else there is a wheel with center v_2 or m respectively. Furthermore $b \in V^r$, else there is a $3PC(b, m)$. When node x_j is strictly closer to x_n than x_i on the path Q, b is adjacent to z, else there is a $3PC(b, z)$. Node m is adjacent to z, else there is a $3PC(m, z)$. This yields a path Q of Type m. When node x_i is closer to x_n than x_j on the path Q, or when $x_i = x_j$, the path Q is of Type n.

Case 4.2 Node x_1 is strongly adjacent to Π.

If x_1 is of Type a, f or j[2.1], then there is a wheel with center v_2. If x_1 is of Type c[2.1], with neighbors in $\tilde{P}_1$ and $\tilde{M}$, then there is an odd wheel with center v. If x_1 is of Type d[2.1], with neighbors in $\tilde{P}_1$ and $\tilde{M}$, then there is a $3PC(x_1, z)$.

Part 2 Node x_n is not strongly adjacent to Π.
Let t be the neighbor of x_n in Π.

Case 1 $N(V(\tilde{Q})) \cap \{v_1, v_2, m\} = \emptyset$.

Case 1.1 Node x_1 is not strongly adjacent to Π.

Let b be the neighbor of x_1 in Π.

Case 1.1.1 $b \in V(\tilde{P}_1)$

Then node z is adjacent to v_2, else there is a $3PC(z, v_2)$. The nodes b and t belong to the same side of the bipartition, else there is a $3PC(b, t)$. If $b, t \in V^c$, then they are both adjacent to v_1, else there is a $3PC(b, v_1)$ or a $3PC(t, v_1)$. This yields paths Q of Types g or h.

Case 1.1.2 $b \in V(M) \setminus \{v, m\}$

Then $b \in V^r$, else there is a $3PC(b, v_1)$ or $3PC(b, v_2)$.

If the top path of Π is long, assume w.l.o.g. that t is not adjacent to v_1. Then node z is adjacent to v_2, else there is a $3PC(z, v_2)$. Node t is adjacent to v_2, else there is a $3PC(z, v_1)$. This yields a path Q of Type i.

If the top path of Π is short, then the path Q is of Type d when the side paths are long and the path Q is of Type i when Π has a short side.

Case 1.2 Node x_1 is strongly adjacent to Π.

Case 1.2.1 Node x_1 is of Type a[2.1].

If one side path of Π is short, say P_2, then there is a wheel with center v_2. If the top path is long, there exists a $3PC(x_1, v_1)$ or a $3PC(x_1, v_2)$. So the top path is short and the side paths are long, yielding a path Q of Type f.

Case 1.2.2 Node x_1 is of Type b[2.1].

Let b_1, b_2 be the neighbors of x_1 in P_1 and P_2 respectively. If b_1 is not adjacent to v_1 or b_2 is not adjacent to v_2, there exists a $3PC(z, x_1)$. If b_1 is adjacent to v_1 and b_2 is adjacent to v_2 then $t \in V^r$, else there is a $3PC(t, x_1)$. This yields a connected 6-hole.

Case 1.2.3 Node x_1 is of Type c[2.1].

Assume w.l.o.g. that x_1 has neighbors in $V(\tilde{M})$ and $V(\tilde{P}_2)$. Since x_1 is not adjacent to v_2, there is a $3PC(x_1, v_2)$.

Case 1.2.4 Node x_1 is of Type d[2.1].

There is a $3PC(x_1, z)$.

Case 1.2.5 Node x_1 is of Type f[2.1].

Node x_1 is not adjacent to both v_1, v_2. If x_1 is not adjacent to v_1, there is a $3PC(x_1, v_1)$. If x_1 is not adjacent to v_2, there is a $3PC(x_1, v_2)$.

Case 1.2.6 Node x_1 is of Type j[2.1].

Assume w.l.o.g. that z is adjacent to v_2. This implies that t is not adjacent to v_2, else there is an odd wheel. Then $t \in V^r$, else there is a $3PC(t, v_2)$. This yields a path Q of Type j.

Case 2 $N(V(\tilde{Q})) \cap \{v_1, v_2, m\} = \{m\}$.

Case 2.1 Node x_1 is not strongly adjacent to Π.

Let b be the neighbor of x_1 in Π.

Case 2.1.1 $b \in V(\tilde{P}_1)$.

Node $b \in V^r$, else there is a $3PC(m, b)$ and node $t \in V^c$, else there is a $3PC(v, t)$. This implies the existence of a $3PC(b, t)$.

Case 2.1.2 $b \in V(M) \setminus \{v, m\}$.

The set $V(Q) \cup \{b\}$ contains at most two nodes adjacent to m, otherwise there is a wheel. If it contains only one neighbor of m, say x_i, there is a $3PC(x_i, v_1)$. So, $V(Q) \cup \{b\}$ contains exactly two neighbors of m. If b is not one of them, there is a parachute modification of Type 4 at the bottom. If the top path of Π is short, this yields a path Q of Type e. If the top path of Π is long, assume w.l.o.g. that t is not adjacent to v_1. Node $t \in V^c$, else there is a $3PC(v, t)$. This implies that v_2 is adjacent to z, otherwise there is a $3PC(t, v_1)$. Since v_1 is not adjacent to z, then t is adjacent to v_2, else there is a $3PC(t, v_2)$. This yields a path Q of Type k.

Case 2.2 Node x_1 is strongly adjacent to Π.

Case 2.2.1 Node x_1 is of Type a [2.1].

There is a parachute modification of Type 4 at the bottom.

Case 2.2.2 Node x_1 is of Type b or d[2.1].

There is a $3PC(x_1, z)$.

Case 2.2.3 Node x_1 is of Type c [2.1].

Node x_1 is adjacent to m, else there is a $3PC(x_1, m)$. But now there is a wheel with center m, since x_1 is not adjacent to v_1 or v_2.

Case 2.2.4 Node x_1 is of Type f [2.1].

Node x_1 is adjacent to m, else there is a $3PC(x_1, m)$. Let n_1 and n_2 be the neighbors of x_1 in P_1 and P_2 respectively. Assume w.l.o.g. that $n_2 \neq v_2$. If $n_1 \neq v_1$, there is a parachute modification of Type 2 at the bottom. Finally t is the neighbor of v_1 in T, otherwise there is a wheel with center m. This yields a path Q of Type a.

Case 2.2.5 Node x_1 is of Type j [2.1].

There is a $3PC(x_1, m)$, since by definition of Q, node x_1 is not adjacent to m.

Case 3 $N(V(\tilde{Q})) \cap \{v_1, v_2, m\} = \{v_1\}$.

Node t is adjacent to v_1, otherwise there is a parachute modification of Type 2 at the top or a $3PC(t, v_1)$.

Case 3.1 Node x_1 is not strongly adjacent to Π.

Let b be the neighbor of x_1 in Π.

Case 3.1.1 $b \in V(P_1) \setminus \{v_1\}$.

If b is adjacent to v_1, there is a wheel with center v_1. Otherwise, there is a parachute modification of Type 3 at the bottom or a $3PC(b, v_1)$.

Case 3.1.2 $b \in V(\tilde{M}) \setminus \{m\}$.

There is a wheel with center v_1.

Case 3.1.3 $b \in V(\tilde{P}_2)$.

Node b is adjacent to v_2, else there is a wheel with center v_1. This yields a $3PC(b, v_1)$.

Case 3.2 Node x_1 is strongly adjacent to Π.

Case 3.2.1 Node x_1 is of Type a[2.1].

If P_2 is long, there is a wheel with center v_1. If P_2 is short, there is a wheel with center v_2.

Case 3.2.2 Node x_1 is of Type b, c or d[2.1].

There is a wheel with center v_1.

Case 3.2.3 Node x_1 is of Type f [2.1].

If x_1 is adjacent to v_1, there is a wheel with center v_1. If x_1 is not adjacent to v_1, there is a $3PC(x_1, v_1)$.

Case 3.2.4 Node x_1 is of Type j [2.1].

If x_1 is adjacent to v_1, there is a wheel with center v_1. If x_1 is adjacent to v_2, then there is a parachute modification of Type 3 at the bottom, a contradiction.

Case 4 $N(V(\tilde{Q})) \cap \{v_1, v_2, m\} = \{v_2, m\}$.

As a consequence of Claim 2, the parachute Π has short side P_2. Let $x_j \in V(\tilde{Q})$ be the neighbor of v_2 closest to t in Q and $x_i \in V(\tilde{Q})$ the neighbor of m closest to t. Note that $i = j$ is possible.

If x_j is strictly closer to t than x_i, then either there is a parachute modification of Type 2 at the top (when $t \in V^r$) or there is a $3PC(v_2, t)$ or a wheel with center v_2 (when $t \in V^c$).

If $x_i = x_j$ and $t \in V^c$, then there is a $3PC(v_2, t)$ or an odd wheel with center v_2. So $t \in V^r$.

If x_i is strictly closer to t than x_j and there is no other neighbor of m on the subpath of Q connecting t to x_j, then there is a $3PC(v_2, x_i)$. So there are two neighbors of m on the subpath of Q connecting t to x_j, say x_i and x_k. Furthermore, $t \in V^c$, else there is a $3PC(v, t)$, and t is adjacent to v_2, else there is a $3PC(v_2, t)$.

Case 4.1 Node $t \in V^r$ and $x_i = x_j$.

Case 4.1.1 Node x_1 is not strongly adjacent to Π.

Let b be the neighbor of x_1 in Π.

Case 4.1.1.1 $b \in V(\tilde{P}_1)$.

There is a $3PC(x_i, t)$ unless the path Q contains a neighbor x_k of v_2 which is distinct from x_i. Now there is a wheel with center v_2 unless the path Q contains a neighbor x_l of m which is distinct from x_i. Note that if $l < k$, then there is again a $3PC(x_i, t)$. So we must have $l \geq k$. If $b \in V^c$, there is a $3PC(b, m)$. This yields a path Q of Type o.

Case 4.1.1.2 $b \in V(M) \setminus \{v, m\}$.

If v_2 is adjacent to a node of $V(Q)$ distinct from x_i, there is a wheel with center v_2. If $b \in V^r$, then either m is adjacent to a node of $V(Q)$ distinct from x_i and there is a parachute modification of Type 4 at the bottom, or m is only adjacent to x_i in $V(Q)$ and there is a $3PC(x_i, b)$. If $b \in V^c$, then b must be adjacent to m, else there is a $3PC(b, m)$. This yields a path Q of Type p.

Case 4.1.2 Node x_1 is strongly adjacent to Π.

If x_1 is of Type a[2.1], then either m is adjacent to a node of $V(Q)$ distinct from x_i and there is a parachute modification of Type 4 at the bottom, or m is only adjacent to x_i in $V(Q)$ and there is a wheel with center v_2.

If x_1 is of Type c[2.1], then there is a wheel with center v.

If x_1 is of Type d[2.1], then there is a $3PC(x_1, z)$.

If x_1 is of Type f[2.1], then there is a parachute modification of Type 2 at the bottom.

If x_1 is of Type j[2.1], then there is a wheel with center v_2.

Case 4.2 Node $t \in V^c$ is adjacent to v_2, node m is adjacent to x_i, x_k on Q and v_2 is adjacent to x_j, where $j \leq k < i$.

Case 4.2.1 Node x_1 is not strongly adjacent to Π.

Let b be the neighbor of x_1 in Π.

If $b \in V(\tilde{P}_1)$, then $b \in V^r$, else there is a $3PC(b, m)$. This yields a path Q is of Type q.

If $b \in V(M) \setminus \{v, m\}$, then there is a wheel with center v_2.

Case 4.2.2 Node x_1 is strongly adjacent to Π.

If x_1 is of Type a[2.1], then there is a wheel with center v_2.

If x_1 is of Type c[2.1], then there is a $3PC(x_1, m)$ if x_1 is not adjacent to m and a wheel with center m if x_1 is adjacent to m.

If x_1 is of Type d[2.1], then there is a $3PC(x_1, z)$.

If x_1 is of Type f or j[2.1], then there is a wheel with center v_2. □

Corollary 3.4 *In a wheel-free balanced bipartite graph, all direct connections described in Theorem 3.3 can exist, except for the connected 6-hole.*

In the remainder of the paper, we consider a wheel-free bipartite graph G that is signable to be balanced and contains no connected 6-hole.

4 Connections from Bottom to Top

In this section, we continue the study of direct connections Q from bottom to top of a parachute. These connections were considered in Theorem 3.3 under the assumption that all possible parachute modifications relative to $V(Q)$ had been performed. Here we describe the possible direct connections before parachute modifications are performed.

Theorem 4.1 *Let G be a wheel-free bipartite graph that is signable to be balanced and contains no connected 6-hole. Let $\Pi = Par(P_1, P_2, M, T)$ be a parachute and let $Q = x_1, \ldots, x_n$ be a direct connection from bottom to top avoiding $S(\Pi)$.*

(i) If Π has long top and long sides, then $n \geq 2$ and Q is of Type a, b or c[3.3] or of one of the following types, see Figure 4.

- **Type a1** *Node x_1 is a strongly adjacent node to Π, adjacent to v_1, m and some node $b \in V(\tilde{P}_2)$. Node x_n is strongly adjacent to Π, adjacent to v_1 and to $t \in V(\tilde{T})$. Exactly one of the nodes x_j, for $2 \leq j \leq n-1$, is adjacent to m and none is adjacent to v_1, v_2.*

- **Type b1** *Node x_n is a strongly adjacent node to Π, adjacent to v_2, m and some node $t \in V(\tilde{T})$. Node x_1 is strongly adjacent to Π, adjacent to v_2 and to $b \in V(\tilde{P}_2)$. For $2 \leq j \leq n-1$, node x_j has no neighbor in Π. Furthermore, Π has a short middle path.*

- **Type b2** *Node x_n is a strongly adjacent node to Π, adjacent to v_2, m and some node $t \in V(\tilde{T})$. Node $x_1 \in V^c$ is not strongly adjacent to Π and its unique neighbor belongs to $V(\tilde{P}_2)$. Exactly one of the nodes x_j, for $2 \leq j \leq n-1$, is adjacent to v_2 and none is adjacent to v_1, m. Furthermore, Π has a short middle path.*

- **Type b3** *Node* x_n *is a strongly adjacent node to* Π*, adjacent to* v_2, m *and some node* $t \in V(\tilde{T})$*. Node* $x_1 \in V^r$ *is strongly adjacent to* Π*, adjacent to* $b_1, b_2 \in V(P_2)$*. Exactly one of the nodes* x_j*, for* $2 \leq j \leq n-1$*, is adjacent to* v_2 *and none is adjacent to* v_1, m*. Furthermore,* Π *has a short middle path.*

(ii) If Π *has short top and long sides, then either* $n = 1$ *and the only node of* Q *is of Type o[2.1], or* $n \geq 2$ *and* Q *is of Types a, d, e or f[3.3] or of one of the following types. See Figure 4.*

- **Type d1** *Node* $x_1 \in V^r$ *is strongly adjacent to* Π *and its two neighbors both belong to* $V(M) \setminus \{v, m\}$*. Node* x_n *is not strongly adjacent to* Π*. For* $2 \leq j \leq n-1$*, node* x_j *has no neighbor in* Π*.*

- **Type e1** *Node* $x_1 \in V^c$ *is not strongly adjacent to* Π *and its unique neighbor, say* b*, belongs to* $V(\tilde{M}) \setminus \{m\}$*. Node* x_n *is not strongly adjacent to* Π*. Node* m *is adjacent to exactly two of the nodes* x_j, x_k*, for* $2 \leq j < k \leq n-1$*. Nodes* v_1, v_2 *are not adjacent to* $V(Q)$*.*

- **Type e2** *Node* $x_1 \in V^r$ *is strongly adjacent to* Π *and its two neighbors both belong to* $V(M) \setminus \{v, m\}$*. Node* x_n *is not strongly adjacent to* Π*. Node* m *is adjacent to exactly two nodes* x_j, x_k*, for* $2 \leq j < k \leq n-1$*. Nodes* v_1, v_2 *are not adjacent to* $V(Q)$*.*

- **Type e3** *Node* x_1 *is a strongly adjacent of Type a[2.1]. Node* x_n *is not strongly adjacent to* Π*. Node* m *is adjacent to exactly two nodes* x_j, x_k*, for* $2 \leq j < k \leq n-1$*. Nodes* v_1, v_2 *are not adjacent to* $V(Q)$*.*

- **Type e4** *Node* x_1 *is a strongly adjacent of Type f adjacent to* $m, b_1 \in V(\tilde{P}_1)$ *and* $b_2 \in V(\tilde{P}_2)$*. Node* x_n *is not strongly adjacent to* Π*. Node* m *is adjacent to exactly one of the nodes* x_k*, for* $2 \leq k \leq n-1$*. Nodes* v_1, v_2 *are not adjacent to* $V(Q)$*.*

(iii) If Π *has a short side and* G *contains no parachute with long sides, then* Π *has short top and either* $n = 1$ *and the only node of* Q *is of Type l[2.1], or* $n \geq 2$ *and* Q *is of Type g[3.3] or as described below. See Figure 4.*

- **Type g1** *Nodes* x_n *is not strongly adjacent to* Π *and its neighbor is the unique node* $t \in V(\tilde{T})$*. Node* x_1 *is strongly adjacent to* Π *and has neighbors* v_1 *and* $b \in V(\tilde{P}_1)$*. For* $2 \leq j \leq n-1$*, node* x_j *has no neighbor in* Π*.*

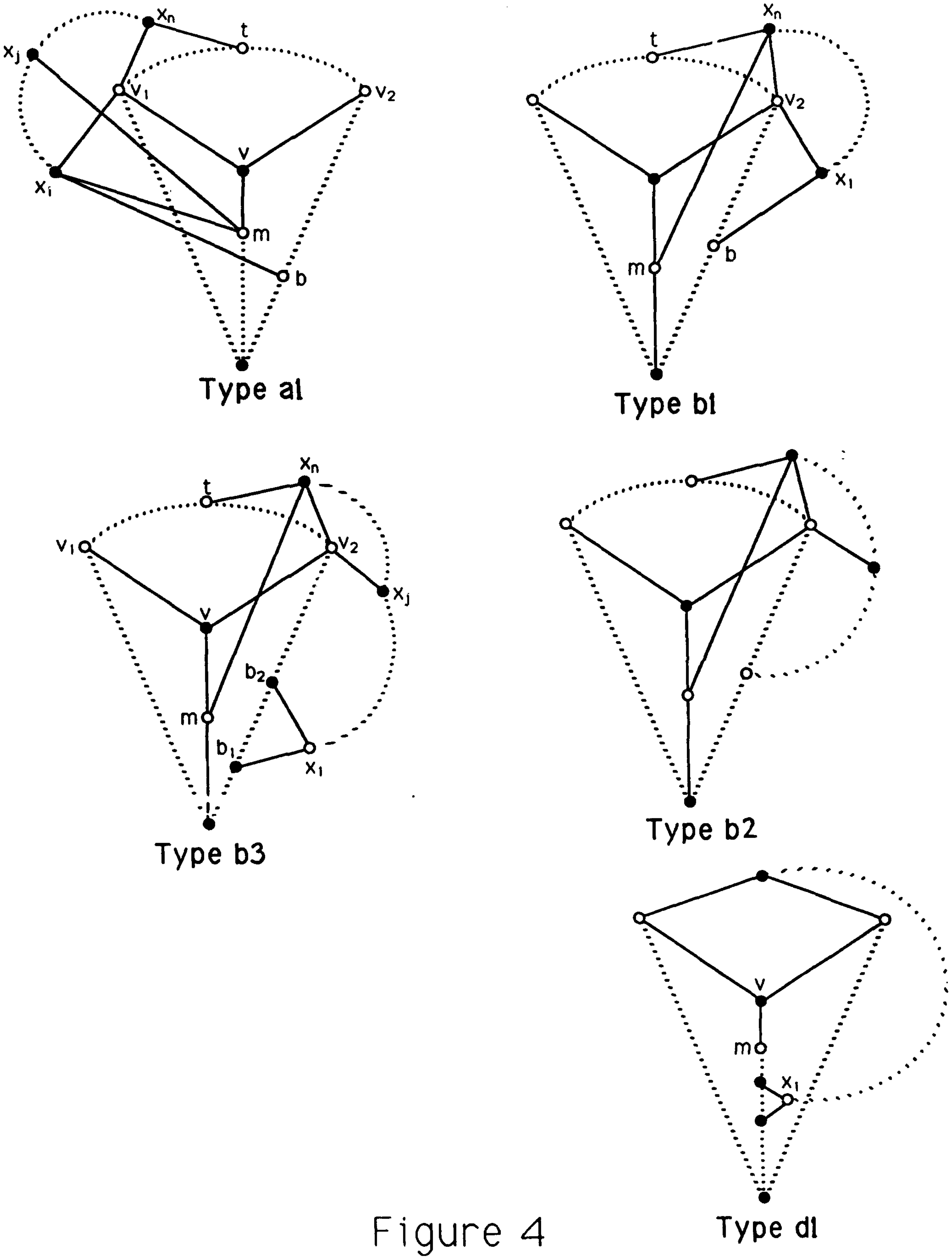
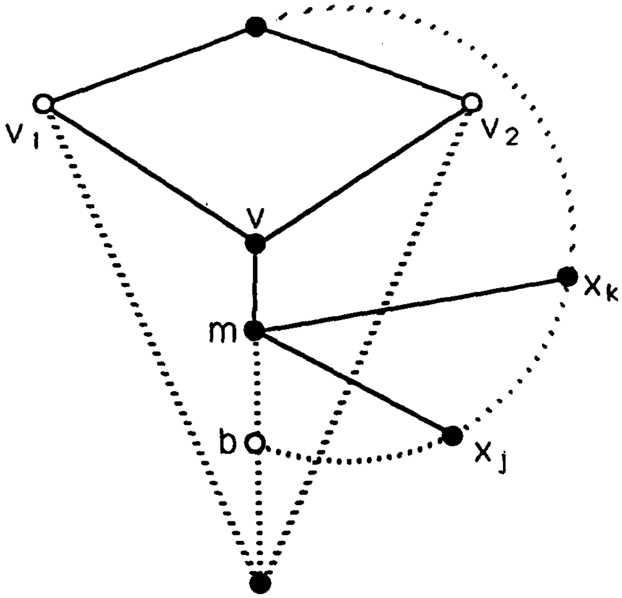
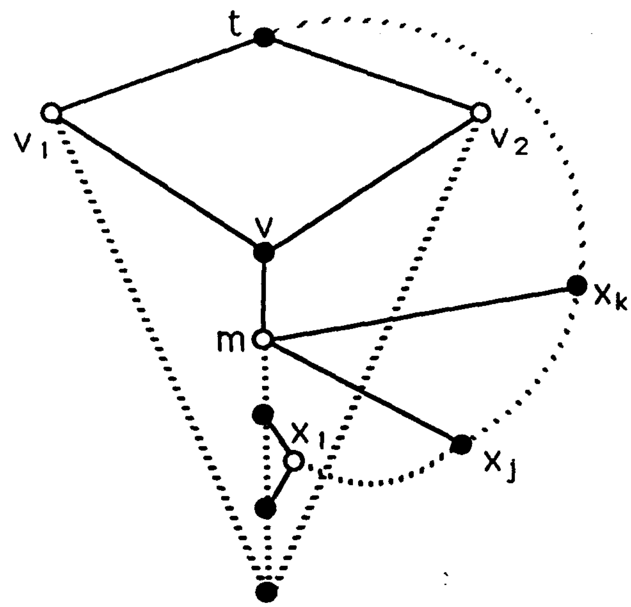


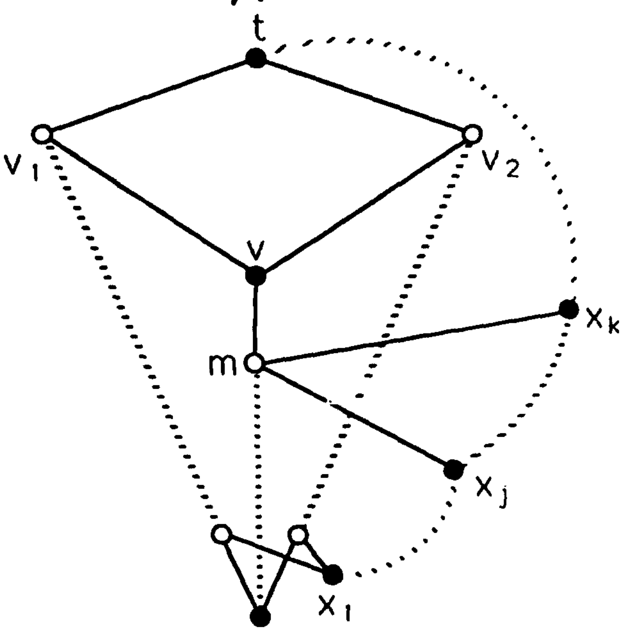
Figure 4



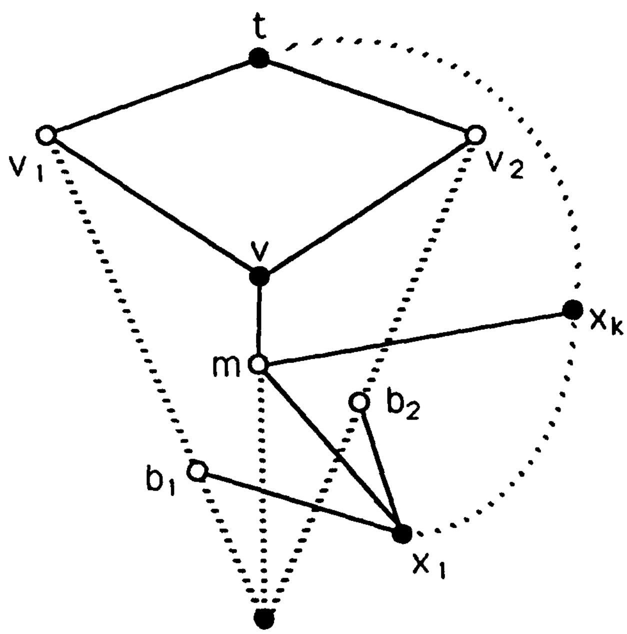
Type e1



Type e2



Type e3



Type e4

Figure 4 (cont.)

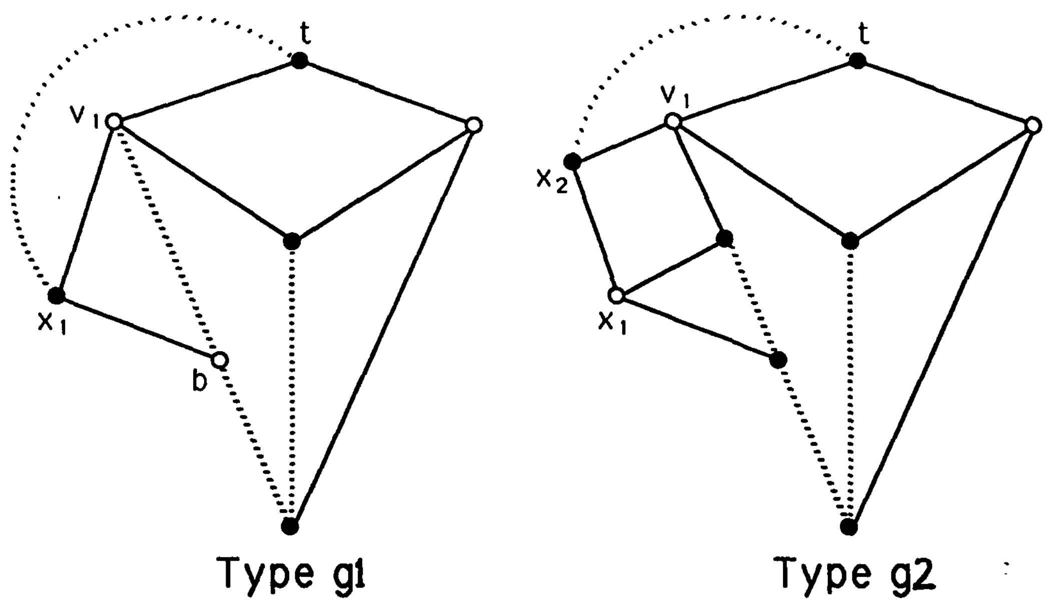


Figure 4: More direct connections from bottom to top

- **Type g2** *Nodes x_n is not strongly adjacent to* Π *and its neighbor is the unique node $t \in V(\tilde{T})$. Node x_1 is strongly adjacent to* Π *and has exactly two neighbors in $V(\tilde{P}_1)$. Furthermore, one of these neighbors is adjacent to v_1. Node x_2 is adjacent to v_1 and, for $3 \leq j \leq n-1$, node x_j has no neighbor in* Π.

Proof: We denote by Π^i, Q^i a parachute and a direct connection such that exactly i parachute modifications must be performed relative to $V(Q^i)$ in order to obtain a parachute and a direct connection described in Theorem 3.3. With this notation, Theorem 3.3 describes all pairs Π^0, Q^0. To prove Theorem 4.1, we describe all pairs Π^i, Q^i that give rise to a pair Π^{i-1}, Q^{i-1}. In this proof we will show that at most two parachute modifications can be performed relative to the nodes of a direct connection, that is, $i \leq 2$.

Our proof uses two properties that follow from the definition of parachute modifications:

Property 4.2 Let Π^{i-1}, Q^{i-1} be obtained from Π^i, Q^i by a parachute modification relative to $V(Q^i)$. Let $Q^{i-1} = x_1, \ldots, x_n$. The nodes x_1 and x_n have exactly one neighbor in $V(\Pi^{i-1}) \setminus \{v_1, v_2, m\}$.

Property 4.3 Let Π^{i-1}, Q^{i-1} be obtained from Π^i, Q^i by a parachute modification at the bottom relative to $V(Q^i)$. Let $Q^{i-1} = x_1, \ldots, x_n$. If $x_1 \in V^c$, then the parachute modification is of Type 1.

(i) By Theorem 3.3(i), $Q^0 = x_1, \ldots, x_n$ must be of Type a, b or c[3.3] relative to Π^0.

Case 1 Path Q^0 is of Type a[3.3] relative to Π^0.

If Π^1, Q^1 has a parachute modification of Type 1 at the top, then Π^1, Q^1 is of Type a1.

Assume Π^1, Q^1 has a parachute modification of Type 2 at the top. Then $\tilde{Q}^1$ contains a node adjacent to v_1 and another adjacent to m, namely, t and x_j respectively, using the notation of Theorem 3.3, Type a, applied to Π^0, Q^0. Now the tx_j-subpath of $\tilde{Q}^1$ together with the nodes of Π^1 induce an odd wheel with center v. So no parachute modification of Type 2 occured at the top.

In Π^0, Q^0, node x_1 is adjacent to v_1, m and $b \in V(\tilde{P}_2)$. If Π^0, Q^0 was obtained from Π^1, Q^1 by a parachute modification at the bottom, then the modification was necessarily of Type 1 and b must have been in $V(Q^1) \setminus V(\Pi^1)$ since v_1 and m remain unchanged. But then x_1 is a strongly adjacent node which violates Theorem 2.1 relative to Π^1.

The above proof shows that Π^2, Q^2 must yield Π^1, Q^1 of Type a1 after one parachute modification. Assume Π^2, Q^2 has a parachute modification of Type 1 or Type 2 at the top. Then $\tilde{Q}^2$ contains a node adjacent to v_1 and another adjacent to m. This shows the existence of an odd wheel with center v. A parachute modification at the bottom cannot occur either by the above argument.

Case 2 Path Q^0 is of Type b[3.3].

Π^0, Q^0 cannot be obtained from Π^1, Q^1 by a parachute modification at the top, since otherwise node x_n would violate Theorem 2.1 relative to Π^1. If Π^0, Q^0 is obtained by a parachute modification of Type 1 at the bottom, then Π^1, Q^1 is of Type b1. If it is obtained by a parachute modification of Type 3 , then Π^1, Q^1 is of Type b2.

If Π^1, Q^1 is of Type b1 or Type b2 and is obtained from Π^2, Q^2 by a parachute modification, then Π^2, Q^2 is of Type b3. Now Property 4.2 shows that Π^2, Q^2 cannot be obtained by any parachute modification .

Case 3 Path Q^0 is of Type c[3.3].

No parachute modification at the bottom or the top can occur, else node x_1 or x_n violates Theorem 2.1 in Π^1.

(ii) Since Π has short top, no parachute modification was performed at the top. By Theorem 3.3, Q^0 must be of Type o[2.1] or of Type a, d, e or f[3.3] relative to Π^0.

Case 1 Path Q^0 is of Type o[2.1] or Type a[3.3].

No parachute modification can be performed at the bottom since node x_1 in Q^0 would violate Theorem 2.1 with respect to Π^1.

Case 2 Path Q^0 is of Type d[3.3].

If Π^0, Q^0 is obtained by a parachute modification of Type 1 at the bottom, then Π^1, Q^1 is of Type d1. Property 4.2 shows that no further parachute modification can occur.

Case 3 Path Q^0 is of Type e[3.3].

If Π^0, Q^0 is obtained by a parachute modification of Type 1 at the bottom, then Π^1, Q^1 is of Type e1, where $x_j = x_1$. If Π^0, Q^0 is obtained by a parachute modification of Type 2 at the bottom, then Π^1, Q^1 is of Type e2. If Π^0, Q^0 is obtained by a parachute modification of Type 4 at the bottom, there are two cases: If x_1 in Q^1 is not strongly adjacent to Π^1, then Π^1, Q^1 is of Type e1. If x_1 is a strongly adjacent node to Π^1 of Type a[2.1], then Π^1, Q^1 is of Type e3.

If Π^1, Q^1 of Type e1 is obtained from Π^2, Q^2 by a parachute modification of Type 1, then Π^2, Q^2 is of Type e4 and Property 4.3 shows that no other parachute modification can be performed.

Property 4.2 shows that Π^1, Q^1 of Type e2, e3 and e4 cannot be obtained from Π^2, Q^2.

Case 4 Path Q^0 is of Type f[3.3]. There cannot be any parachute modification.

(iii) By Theorem 3.3(iii), the path Q^0 is either a single strongly adjacent node of Type k, l, m or n[2.1] or a path of Type b, g, h, i, j, k, l, m, n, o, p or q (Theorem 3.3). It is easy to check that each of the configurations k, m or n[2.1] and b, h, i, j, k, l, m, n, o, p or q[3.3] contains a parachute with long sides as an induced subgraph. This leaves only two cases: Type l[2.1] and Type g[3.3]. In both cases, there is a parachute with long sides, unless the top path of Π^0 is short.

Now consider the case where one or more parachute modifications occured. Since a short top in Π^0 cannot arise by path modification at the top, Π^1 and Π^0 have the same top path, and therefore Π^1 has a short top. Now consider the parachute modifications at the bottom that can give rise to Π^0, Q^0. They are either of Type 1 or 3. If a parachute modification of Type 3 was performed and the first node x_1 of Q^1 is not strongly adjacent to Q^0, then there is a parachute with long sides (the center node is v_1, the top path is Q^0). If the first node x_1 of Q^1 is strongly adjacent, then it is of Type j[2.1], adjacent to v_2 and to a node in $V(M)$. In this case there is a parachute with long sides having v_2 as center and T as middle path.

If a single parachute modification of Type 1 was performed and the neighbors of x_1 both belong to $V(\tilde{P}_1)$ in Π^0, then there is a parachute with long sides (the center node is x_1 and the middle path is Q^1). This yields Type g1. If two parachute modifications of Type 1 were performed, then one of the neighbors of x_1 must be adjacent to v_1, else there is a parachute with long sides. This yields Type g2. □

5 Parachutes with a Short Side

As in the earlier section, G is a wheel-free bipartite graph which is signable to be balanced and contains no connected 6-hole. We show that, if G contains a parachute with one short side but no parachute with long sides, then G has an extended star cutset or contains an R_{10} configuration, as defined in the introduction.

Theorem 5.1 *Let G be a wheel-free bipartite graph which is signable to be balanced and contains no parachute with long sides. Let $\Pi = Par(P_1, P_2, M, T)$ be a parachute with a short side, say $P_2 = v_2, z$ and let its middle path be $M = v, m, \dots, z$. Then at least one of the following alternatives holds:*

- *The set $S(\Pi)$ is an extended star cutset of G.*
- *The set $N(v_2) \cup (N(z) \cap N(v)) \setminus \{m\}$ is an extended star cutset of G.*
- *The graph G contains an R_{10} configuration.*

Proof: Since G contains no parachute with long sides, G contains no connected 6-hole. If $S(\Pi)$ is not an extended star cutset then, by Theorem 4.1(iii), Π has short top v_1, t, v_2 and, after possibly a parachute modification at the bottom, there is a direct connection from bottom to top of Type g[3.3] or of Type l[2.1]. Denote by Π', Q' the parachute and direct connection after parachute modification, if any. The parachute Π' is identical to Π except possibly for path P_1 which is modified into P_1'. Note that Π' induces another parachute with short side, namely the parachute with center node v_2, side nodes v, z, top path M, middle path T and side paths P_1' and $P_2' = v, v_1$. Denote by Π^* this parachute. By definition, $S(\Pi^*) = N(v_2) \cup (N(z) \cap N(v)) \setminus \{m\}$. If $S(\Pi^*)$ is not an extended star cutset then, by Theorem 4.1(iii), Π^* has short top v, m, z and, there is a direct connection R' from the bottom of Π^* to the top m of Type l[2.1], Type g[3.3] , Type g1 or g2[4.1]. Assume first that R' is a direct connection of Type g2 and its first node y_1' is adjacent to v_1. Node y_2' is adjacent to z. No node y_k', $k > 2$, is adjacent to a node in Q' else there is a direct connection violating Theorem 4.1. This implies the existence of a wheel with center z. Hence the first node y_1' of R' is not adjacent to v_1. This shows that if R' is of Type g2, a parachute modification can be performed at the bottom without changing the neighbor of Q' on P_1'.

Similarly, for Type g1, a parachute modification can be performed at the bottom without changing the neighbor of Q' on P_1'. Let P_1'' be the corresponding modification of P_1', if any. Denote by Π'' the parachute obtained from Π by replacing P_1 by P_1''. The endpoints of Q', say x_n' and x_1', are adjacent to t and to the neighbor a of v_1 on P_1'' respectively, and the endpoints of R', say y_k' and y_1' are adjacent to m and to the neighbor b of z on P_1''. See Figure 5.

To complete the proof of the theorem, we show the following result.

Claim: $V(\Pi'') \cup V(Q') \cup V(R')$ induces an R_{10} configuration.

Proof of Claim: If Q' and R' have no common or adjacent node, then there is an odd wheel. If any node of Q' other than x_1' is adjacent to or coincident with a node of $V(R')$, then there is a direct connection from bottom to top which is not of Type g[3.3] or Type g1, g2[4.1], a contradiction. Similarly, the only node of R' that can be adjacent to or coincident with a node of $V(Q')$ is y_1'. So assume x_1' is adjacent to y_1'. Node x_1' is adjacent to t, else there is a $3PC(x_1', t)$. Similarly, y_1' is adjacent to m, else there is a $3PC(y_1', m)$. Finally, a is adjacent to b, else there is a $3PC(a, b)$. But now we have the configuration R_{10} (see Part I for the definition). This completes the proof of

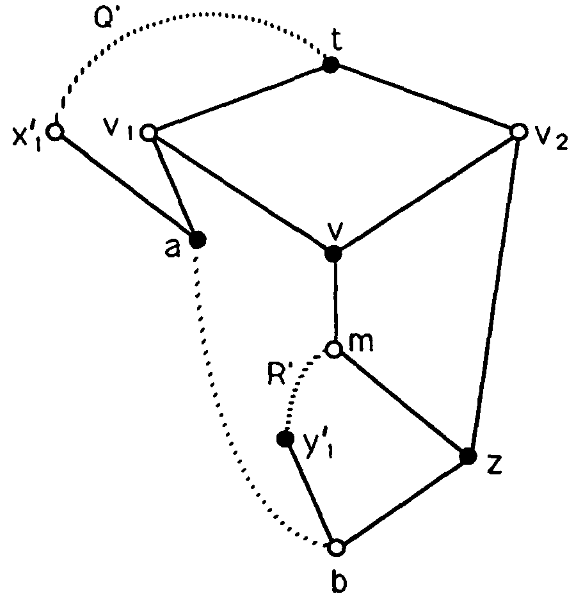


Figure 5: Parachute Π''

the claim and of the theorem. □

6 Stabilized Parachutes

In the remainder of this part, we consider bipartite graphs G which contain a parachute with long sides. As in the earlier sections, we assume that G is wheel-free, signable to be balanced and contains no connected 6-hole. In this section, we make the further assumption that G contains a stabilized parachute, as defined below. See Figure 6.

Definition 6.1 *A* stabilized parachute (Π, R) *consists of a parachute* $\Pi = Par(P_1, P_2, M, T)$ *with long side paths* $P_1 = v_1, a, \ldots, z$ *and* $P_2 = v_2, \ldots, z$*, a short middle path* $M = v, m, z$ *and of a chordless path* $R = r_1, \ldots, r_k$*,* $k \geq 1$*, where* $r_i \in V \setminus V(\Pi)$ *for* $i = 1, \ldots, k$*, such that node* r_1 *is adjacent to node* a *and node* r_k *is adjacent to* v*. Nodes* r_1 *and* r_k *do not have any other adjacencies in* Π *than those just mentioned and nodes* r_i *for* $i = 2, \ldots, k-1$*, are not adjacent to any node of* Π*. Furthermore,*

(i) any strongly adjacent node of Type f[2.1] relative to Π *which is adjacent to* v_2 *must also be adjacent to* v_1*, and*

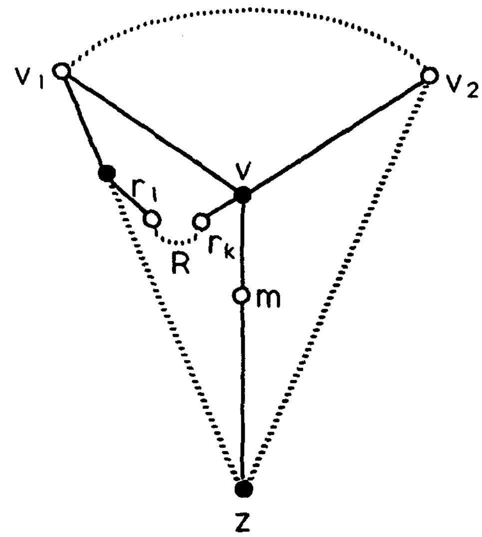


Figure 6: Stabilized parachute

(ii) any node in $V \setminus (V(\Pi) \cup V(R))$ which has two neighbors in $V(T)$ and is adjacent to r_k must also be adjacent to m.

In this section we prove that if G contains a stabilized parachute, then G has an extended star cutset. It follows as a corollary that if G contains a parachute with long top and long sides, then G has an extended star cutset.

Lemma 6.2 *If G contains a parachute Π with long sides having a direct connection of Type a, b or c[3.3] or Type a1, b1, b2 or b3[4.1], then G contains a stabilized parachute.*

Proof: We divide the proof in the following cases:

Case 1 $Q = x_1, \ldots, x_n$ is a direct connection of Type a[3.3].

Assume w.l.o.g. that Q is a shortest direct connection of Type a[3.3] and let x_j be the intermediate node of Q adjacent to m. We use the notation of Figure 3.

Case 1.1 There exists a node w adjacent to two nodes x_f and x_i in the set $\{x_1, \ldots, x_j\}$ and to the neighbor m' of m in $V(M) \setminus \{v\}$ but not adjacent to v.

Assume w.l.o.g. that $i > f$. Then the path $S = w, x_i, x_{i+1}, \ldots, x_n$ is a direct connection from bottom to top in Π and Theorem 4.1 shows that S must be a direct connection of Type e[3.3]. Hence the top path T of Π must be short.

If $x_f \neq x_1$, the hole $w, x_i, x_{i+1}, \ldots, x_n, t, v_1, x_1, b, \ldots, z, \ldots, m', w$ induces an odd wheel with center m.

If $x_f = x_1$, consider the extended parachute (Π", R) having node x_1 as center, nodes w and b as side nodes and x_1, v_1, t as middle path. The path $R = x_2, \ldots, x_{i-1}$.

No node of Type f[2.1] relative to Π" is adjacent to b, v_1 and a node x_l, $i \leq l \leq n$. For, if such a node exists, then it must also be adjacent to m, otherwise there is a direct connection violating Theorem 4.1 in Π. However this implies the existence of a direct connection of Type a[3.3] which is shorter than Q, contradicting the assumption. Hence Condition 6.1(i) is satisfied by (Π", R).

No node u can be adjacent to x_2 and to two nodes in the top path of Π", otherwise either u violates Theorem 2.1 in Π" or Π has a direct connection violating Theorem 4.1. Hence Condition 6.1(ii) is satisfied by (Π", R).

Case 1.2 Every node adjacent to two nodes in the set $\{x_1, \ldots, x_j\}$ and to the neighbor of m in $V(M) \setminus \{v\}$ is also adjacent to v.

Then a stabilized parachute occurs from $V(Q) \cup V(\Pi) \setminus V(\tilde{P}_1)$ by taking m as the center node, x_1, x_j as the side nodes and v_2 as the bottom node. The node v_1 is of Type f[2.1] relative to this parachute and forces Condition (i) to hold in Definition 6.1 (otherwise there is an odd wheel). Finally, Condition (ii) of Definition 6.1 holds for (Π', R) by assumption.

Case 2 $Q = x_1, \ldots, x_n$ is a direct connection of Type b[3.3].

Then, using the notation of Theorem 3.3, a stabilized parachute occurs from $V(Q) \cup V(\Pi)$ by taking v_2 as the center node, b, x_n as the side nodes and v_1 as the bottom node. Node m is of Type f[2.1] relative to this parachute and forces Condition (i) of Definition 6.1 to hold. Finally, assume there is a node w adjacent to two nodes in the set $\{x_1, \ldots, x_n\}$ and to the neighbor of v_2 in $V(T)$. Then, there is a direct connection from bottom to top of Π which contradicts Theorem 4.1, unless w is also adjacent to v. Therefore Condition (ii) of Definition 6.1 holds for (Π', R).

Case 3 $Q = x_1, \ldots, x_n$ is a direct connection of Type c[3.3].

Then, using the notation of Theorem 3.3, a stabilized parachute occurs by taking v_2 as the center node, x_1, x_n as the side nodes and v_1 as the bottom node. The strongly adjacent nodes of Type f[2.1] relative to this parachute are either all adjacent to the side node x_1 or all adjacent to the side node x_n (else there is an odd wheel). Since there are two possibilities for the path R, namely the subpath of T from v_2 to t and the path from v_2 to b, Condition (i) of Definition 6.1 is satisfied by one of the choices for R. Next, we consider Condition (ii). First, consider the case when R is the subpath of T connecting v_2 to t. If there is a node w adjacent to two nodes in the set $\{x_1, \ldots, x_n\}$ and to the neighbor of v_2 in T, then there is a direct connection from bottom to top of Π which contradicts Theorem 4.1, unless w is also adjacent to v. Therefore Condition (ii) holds. Now, consider the case when R connects v_2 to b. If there is a node w adjacent to two nodes in the set $\{x_1, \ldots, x_n\}$ and to the neighbor q of v_2 in P_2, then one of three possibilities occurs.

If w is also adjacent to v, Condition (ii) holds.

If w is not adjacent to v but is adjacent to at least one node of $V(\Pi) \setminus \{v, q\}$, then there is a direct connection from bottom to top of Π which contradicts Theorem 4.1.

If w is not adjacent to any node of $V(\Pi) \setminus \{q\}$, then there is a direct connection of Type b[3.3] from bottom to top of Π, and we have already proved the existence of a stabilized parachute in this case.

Case 4 Q is of Type a1, b1, b2 or b3[4.1].

Then, after parachute modification, a path of Type a or b[3.3] arises and the result has already been estabished above when such a path exists. □

Lemma 6.3 *If Π is a stabilized parachute then the only possible direct connections from bottom to top avoiding $S(\Pi)$ are of Type b or c[3.3] or Type b1, b2 or b3[4.1].*

Proof: A direct connection of Type d[3.3] cannot occur since a stabilized parachute has middle path of length 2. Similarly for Types d1, e1 and e2[4.1]. Now we show that a direct connection $Q = x_1, \ldots, x_n$ of Type a, e or f[3.3] and Type a1, e3 or e4[4.1] cannot occur.

Case 1 Path Q is of Type a[3.3].

It follows from Condition (i) of Definition 6.1 that, if x_1 is adjacent to v_1, m and $b \in V(\tilde{P}_2)$, then r_1 is adjacent to node a, the neighbor of v_1 in P_1. Nodes $x_2, \ldots, x_n$ are not adjacent to any of the nodes $r_1, \ldots, r_{k-1}$, else there is a direct connection from bottom to top which contradicts Theorem 4.1. Node r_k has at most two neighbors in Q, else there exists a wheel with center r_k. We consider three subcases based on the number of neighbors of r_k in $\{x_2, \ldots, x_n\}$.

Case 1.1 Node r_k is not adjacent to any node in the set $\{x_2, \ldots, x_n\}$.

There is a wheel with center v_1 whether or not x_1 is adjacent at least one node in the set $\{r_1, \ldots, r_k\}$.

Case 1.2 Node r_k is adjacent to exactly one node $q \in \{x_2, \ldots, x_n\}$.

If r_k is not adjacent to x_1, then there is a $3PC(v_1, q)$, whether or not x_1 is adjacent to at least one node in $\{r_1, \ldots, r_{k-1}\}$. If r_k is adjacent to x_1, let x_j be the node of $\{x_2, \ldots, x_n\}$ which is adjacent to m. The nodes of Q in the subpath connecting q to x_j together with the nodes $V(R) \cup \{a\} \cup V(T) \cup V(P_2) \cup \{m\}$ induce a wheel with center v.

Case 1.3 Node r_k is adjacent to exactly two nodes $q_1, q_2 \in \{x_2, \ldots, x_n\}$.

Let x_j be the node of $\{x_2, \ldots, x_n\}$ which is adjacent to m and, w.l.o.g. let q_1 be the neighbor of r_k which is closest to x_j. The nodes of Q in the subpath connecting x_j to q_1 together with the nodes $V(R) \cup \{a\} \cup V(T) \cup V(P_2) \cup \{m\}$ induce a wheel with center v.

Case 2 Path Q is of Type a1[4.1].

By Condition (ii) of Definition 6.1, node x_n is not adjacent to r_k. Therefore, after parachute modification, we are back in Case 1.

Case 3 Path Q is of Type e[3.3].

If some node of $V(Q) \setminus \{x_1\}$ is adjacent to at least one node of $V(R) \setminus \{r_k\}$, then there is a direct connection from the bottom to the top of Π different from those listed in Theorem 4.1(ii). So no such adjacency exists. If x_1 is adjacent to two or more nodes of R, there is a wheel with center x_1. If x_1 is adjacent to exactly one node of R, then there is a $3PC(x_1, a)$, where a is the neighbor of v_1 in P_1. So x_1 is not adjacent to a node in R. If r_k is adjacent to a node in Q, then there is a wheel with center v. If r_k is not adjacent to a node in Q, then there is a wheel with center v_1.

Case 4 Path Q is of Type f[3.3] or Type e2 or e3(Theorem 4.1).

No node of $V(Q) \setminus \{x_1\}$ is adjacent to $V(R) \setminus \{r_k\}$ since this would contradict Theorem 4.1(ii). If x_1 is adjacent to a node in R, there is a wheel with center x_1. So x_1 is not adjacent to a node in R. If r_k is adjacent to a node in Q, then there is a $3PC(x_1, r_k)$. If r_k is not adjacent to a node in Q, then there is a wheel with center v_1. $\square$

Theorem 6.4 *If G contains a stabilized parachute, then G has an extended star cutset.*

Proof: Among all parachutes that give rise to a stabilized parachute, let Π be one with shortest top. We will show that $S(\Pi)$ is an extended star cutset, i.e. a path Q of Type a-f[3.3] or a1, b1-b3, d1, e1-e4[4.1] cannot occur. For Types a, d, e and f[3.3] and a1, d1, e1-e4[4.1], the result follows from Lemma 6.3.

Now consider the case where $Q = x_1, \ldots, x_n$ is a direct connection of Type b[3.3] relative to Π. Assume w.l.o.g. that x_1 is adjacent to the neighbor b of v_2 in P_2. Note that the extra path R has its first node r_1 adjacent either to b or to the neighbor a of v_1 in P_1. Construct the parachute Π' as follows. The middle path of Π' is $M' = x_n, m, z$. The top path T' of Π' is the subpath of T connecting the two neighbors of x_n in T, namely t and v_2. The side path P_2' is identical to P_2 and the side path P_1' connects t to z, using nodes of $V(T) \cup V(P_1)$. The new extra path is induced by $\{x_1, \ldots, x_{n-1}\}$. We will show that Π' defines a stabilized parachute with shorter top than Π, contradicting the choice of Π. In order to prove that Π' defines a stabilized parachute, we need to check Conditions (i) and (ii) of Definition 6.1. Condition (i) holds since $t \in V(\tilde{T})$ and a node w of Type f[2.1] relative to Π' which is adjacent to t must also be adjacent to v_2, else w violates Theorem 2.1 relative to Π. To see that Condition (ii) holds, consider a node y adjacent to x_{n-1} and to two nodes of T'. There is a direct connection which violates Theorem 4.1(i) with respect to Π, unless node y is adjacent to m. This completes the proof that Π' is an stabilized parachute.

Now consider the case where $Q = x_1, \ldots, x_n$ is a direct connection of Type b1-b3[4.1]. After parachute modification, we have a direct connection of Type b[3.3] and the argument presented just above can be applied, since the path R plays no role in it.

There only remains the case where the path $Q = x_1, \ldots, x_n$ is of Type c[3.3]. Assume w.l.o.g. that x_1 and x_n are adjacent to v_1. Then the first node r_1 of the extra path $R = r_1, \ldots, r_k$ is adjacent to the neighbor of v_1 in P_1, by Condition (i) of Definition 6.1. Note that the nodes $x_2, \ldots, x_n$ are not adjacent to $r_1, \ldots, r_{k-1}$, else there is a direct connection from bottom to top which contradicts Theorem 3.3.

If r_k is adjacent to at least one node in $\{x_2, \ldots, x_n\}$, then there is a parachute with shorter top path obtained by replacing the center node v by the node x_n and replacing the extra path R by a chordless path from x_n to r_1 only involving nodes of $(V(Q) \setminus \{x_1\}) \cup V(R)$. Condition (i) of Definition 6.1 is satisfied since the new extra path still has r_1 as first node. To see that Condition (ii) holds, consider a node w adjacent to x_{n-1} and to two nodes of the new top. There is a direct connection which violates Theorem 4.1(i) with respect to Π, unless node w is adjacent to m.

If r_k is not adjacent to any node in $\{x_2, \ldots, x_n\}$, there is a wheel with center v_1 whether or not x_1 is adjacent to $V(R)$. □

Corollary 6.5 *If G contains a parachute with long top and long sides, then G has an extended star cutset.*

Proof: Follows from Theorem 4.1(i), Lemma 6.2 and Theorem 6.4. □

7 Parachutes with Short Middle Path

In this section, we assume that G is signable to be balanced but does not contain wheels, connected squares, connected 6-holes and stabilized parachutes. We show that, if G contains a parachute with long sides and short middle, then G has an extended star cutset.

Definition 7.1 *For $k \geq 2$, a k-*parachute *Π^k is defined as follows, see Figure 7. For k even, say $k = 2p$, Π^k consists of nodes $v, m, v_1, \ldots, v_{p+1}, x_1, \ldots, x_{p+1}, y_1, \ldots, y_p, z_1, \ldots, z_p$ and chordless paths P_j, for $j = 1, \ldots, 2p$ where:*

- *node v is adjacent to nodes m and $v_1, \ldots, v_{p+1}$,*
- *node m is adjacent to nodes v and $z_1, \ldots, z_p$,*
- *for $t = 1, \ldots, p+1$, node x_t is adjacent to nodes $v_1, \ldots, v_t$,*

- *for $t = 1, \dots, p$, node y_t is adjacent to nodes $v_1, \dots, v_t$,*
- *for $t = 1, \dots, p$, path P_{2t-1} connects x_t to z_t and path P_{2t} connects y_t to v_{t+1},*
- *for $i \neq j$, $V(P_i) \cap V(P_j) = \emptyset$,*
- *there are no adjacencies between the nodes of $V(\Pi^k)$ other than those indicated above.*

For k odd, say $k = 2p+1$, the k-parachute Π^k is obtained from Π^{k-1} by adding a node z_{p+1} adjacent to m, a node y_{p+1} adjacent to nodes $z_1, \dots, z_{p+1}$ and a chordless path P_{2p+1} connecting x_{p+1} to z_{p+1} whose inner nodes are distinct from $V(\Pi^{k-1}) \cup \{z_{p+1}, y_{p+1}\}$ and are not adjacent to $\{y_{p+1}\} \cup V(\Pi^{k-1}) \setminus \{x_{p+1}\}$.

This definition implies that a 2-parachute is a parachute with long side paths P_1, P_2, short middle path v, m, z_1 and short top path v_1, x_2, v_2.

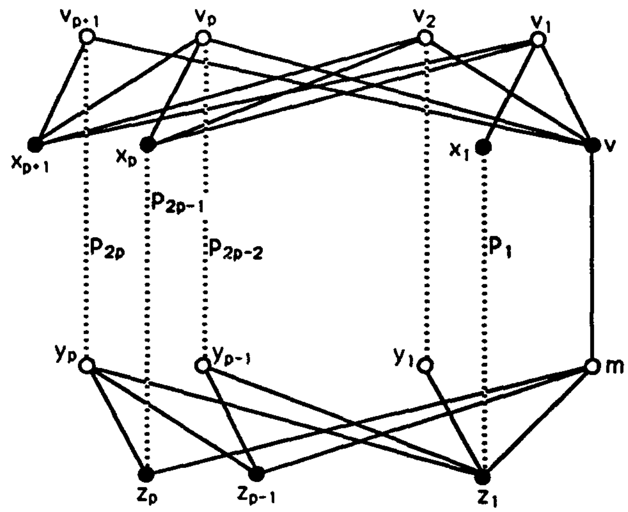
Theorem 7.2 *Assume that G is signable to be balanced but contains no wheel, no connected squares, no connected 6-hole and no stabilized parachute. If G contains a parachute with long sides and short middle, then G has an extended star cutset.*

Proof: Consider a parachute Π with long sides and short middle. If Π has long top, then G has an extended star cutset by Corollary 6.5. So we assume Π has a short top. As noted earlier, Π is a 2-parachute. To establish the theorem, we will prove the following claim.

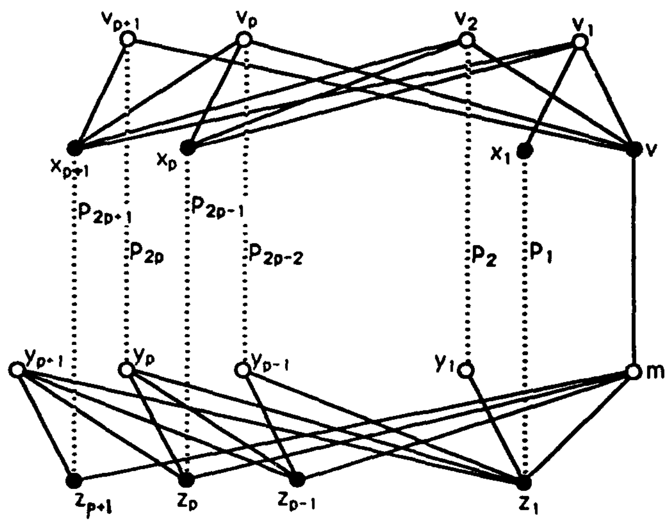
Claim *If G contains a k-parachute, for $k \geq 2$, then either G has an extended star cutset or G contains a $k+1$-parachute.*

Clearly, this claim implies that G has an extended star cutset since G is a finite graph and therefore does not contain arbitrarily large k-parachutes.

Proof of Claim: First, consider the case $k = 2$. The 2-parachute reduces to a parachute Π with long sides, short middle and short top. If $S(\Pi)$ is not an extended star cutset then, by Theorem 4.1(ii), there is a direct connection Q of Type e[3.3], e2 or e3[4.1], since Type o[2.1], Type d[3.3] and Types d1, e1 or e2[4.1] cannot occur when the middle path is short, and Types a or f[3.3] are excluded by the hypothesis of the theorem.



2p-parachute



(2p+1)-parachute

Figure 7:

Case 1 Q is of Type e[3.3].

Using the notation of Theorem 3.3, this configuration of Type e contains another parachute, with center node m, side nodes z, x_j and bottom node v_1. Note that the sides are long. If the top path (connecting z to x_j) is long then, by Corollary 6.5, G has an extended star cutset. If the path connecting z to x_j has only one intermediate node, then the resulting configuration is a 3-parachute.

Case 2 Q is of Type e2 or e3[4.1].

Whether Q is of Type e2 or e3 relative to Π, in each case there are two other parachutes with long top and short middle, say Π_1 and Π_2: using the notation of Theorem 4.1, the center node of Π_1 is m, the middle path is m, v, v_1, one side path Q' is the path connecting x_k to v_1 with nodes in $V(Q) \cup \{t, v_1\}$, the other side path contains P_1; for parachute Π_2, the middle path is m, v, v_2, one side path is Q', the other side path contains P_2. If these parachutes have long top, then G has an extended star cutset by Corollary 6.5. So we assume that Π_1 and Π_2 have a short top. Applying Theorem 4.1(ii) to these parachutes, the only paths R from bottom to top are of Type e[3.3], e2 or e3[4.1]. If the path R is of a certain type relative to Π_1, then it is of the same type relative to Π_2, since the adjacencies between the first node of R and $\{v\} \cup V(\tilde{Q}')$ suffice to distinguish the three possibilities. Now, if R is of Type e2 or e3[4.1], the first node of R is a strongly adjacent node to Π which contradicts Theorem 2.1. So R must be of Type e[3.3] relative to Π_1 and Π_2. Now we are back to Case 1 above. It follows that G contains an extended star cutset or a 3-parachute.

Now, we consider a k-parachute Π^k with $k \geq 3$. First, we assume that k is odd, say $k = 2p + 1$.

In the remainder of the proof, we consider several parachutes. The parachute Π^* is obtained as follows. The middle path is m, v, v_{p+1}, with v_{p+1} as the bottom node; the side nodes are z_p and z_{p+1} and the side paths are P_{2p} and P_{2p+1} respectively; finally the top node is y_{p+1}. Π^* has long sides, short top and short middle. By Theorem 4.1(ii), $S(\Pi^*)$ is an extended star cutset or there is a path P_{2p+2} of Type e[3.3], e2 or e3[4.1] relative to Π^*.

The parachute Π^{**} is the same as Π^* except that the bottom node v_{p+1} is replaced by v_p and the side path P_{2p} is replaced by P_{2p-1}. If the path P_{2p+2}

is of a certain type relative to Π^*, then it is of the same type relative to Π^{**}, since the adjacencies between the first node of P_{2p+2} and $\{v\} \cup V(\tilde{P}_{2p+1})$ suffice to distinghish the three possibilities.

Now consider the parachute Π^{***} with side paths P_{2p-1} and P_{2p}, middle path v, m, z_p and top path v_p, x_{p+1}, v_{p+1}. If P_{2p+2} is of Type e2 or e3 (Theorem 4.1) relative to Π^* and Π^{**}, the first node of P_{2p+2} is a strongly adjacent node to Π^{***} which contradicts Theorem 2.1. So P_{2p+2} must be of Type e[3.3] relative to Π^* and Π^{**}. This implies that $P_{2p+2} = x_{p+2}, v_{p+2}, \ldots, y_{p+1}$, where node x_{p+2} is adjacent to v_p and v_{p+1}, and node v_{p+2} is adjacent to v. There are no other adjacencies between $V(P_{2p+2})$ and $\{v, m\} \cup V(P_{2p-1}) \cup V(P_{2p}) \cup V(P_{2p+1})$. To show that $V(P_{2p+2}) \cup V(\Pi^k)$ induces a $k+1$-parachute, it remains to show that x_{p+2} is adjacent to $v_1, \ldots, v_{p-1}$ and that P_{2p+2} has no common node with or adjacent node to P_j, for $1 \leq j \leq 2p-2$.

Consider the parachute Π_j with middle path m, v, v_j and side paths P_{2j-2} and P_{2p+1}. The top node is y_{p+1}. The path P_{2p+2} connects the bottom of this parachute to the top and avoids $S(\Pi_j)$, since x_{p+2} is adjacent to v_{p+1} and v_{p+1} is adjacent to x_{p+1}, which belongs to the bottom part of Π_j. Since v_{p+1} is not adjacent to any node in P_{2j-2}, the path P_{2p+2} cannot be of Type e2 or e3[4.1] relative to Π_j. Therefore it is a path of Type e[3.3] relative to Π_j, implying that x_{p+2} is adjacent to v_j. Furthermore, there is no common node or adjacency between P_{2p+2} and P_{2j-2}.

Finally, consider the parachute Π_j' with middle path m, v, v_j, side paths P_{2j-1} and P_{2p+1} and top node y_{p+1}. As above, P_{2p+2} is a path of Type e[3.3] relative to Π_j'. It follows that there is no common node or adjacency between P_{2p+2} and P_{2j-2}. Therefore $V(P_{2p+2}) \cup V(\Pi^k)$ induces a $k+1$-parachute.

When $k \geq 4$ is even, the proof is the same as for k odd, interchanging the roles of v and m, v_j and z_j, x_j and y_j. □

8 The Parachute Theorem

Theorem 8.1 *Assume G is a wheel-free bipartite graph which is signable to be balanced but contains no extended star cutset, no connected squares, no connected 6-hole and no R_{10}. Let Π be a parachute with long sides. Then Π has short top and long middle and there exists a direct connection of Type d[3.3] or d1[4.1]. Furthermore, any direct connection from bottom to top avoiding $S(\Pi)$ is of one of these two types.*

Proof: The proof follows by Theorems 4.1, 5.1, Lemma 6.2 and Theorems 6.4 and 7.2. □

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In this seven part paper, we prove the following theorem:

At least one of the following alternatives occurs for a bipartite graph G:

- The graph G has no cycle of length $4k+2$.
- The graph G has a chordless cycle of length $4k+2$.

- There exist two complete bipartite graphs K_1,K_2 in G having disjoint node sets, with the property that the removal of the edges in K_1,K_2 disconnects G.
- There exists a subset S of the nodes of G with the property that the removal of S disconnects G, where S can be partitioned into three disjoint sets T,A,N such that $T \neq \emptyset$, some node $x \in T$ is adjacent to every node in $A \cup N$ and, if $|T| \geq 2$,then $|A| \geq 2$ and every node of T is adjacent to every node of A.

A 0,1 matrix is balanced if it does not contain a square submatrix of odd order with two ones per row and per column. Balanced matrices are important in integer programming and combinatorial optimization since the associated set packing and set covering polytopes have integral vertices.

To a 0,1 matrix A we associate a bipartite graph $G(V^r,V^c;E)$ as follows: The node nets V^r and V^c represent the row set and the column set of A and edge ij belongs to E if and only if $a_{ij}=1$. Since a 0,1 matrix is balanced if and only if the associated bipartite graph does not contain a chordless cycle of length $4k+2$, the above theorem provides a decomposition of balanced matrices into elementary matrices whose associated bipartite graphs have no cycle of length $4k+2$. In Part VII of the paper, we show how to use this decomposition theorem to test in polynomial time whether a 0,1 matrix is balanced.